# Introduction to Cryptography

# Lecture 3

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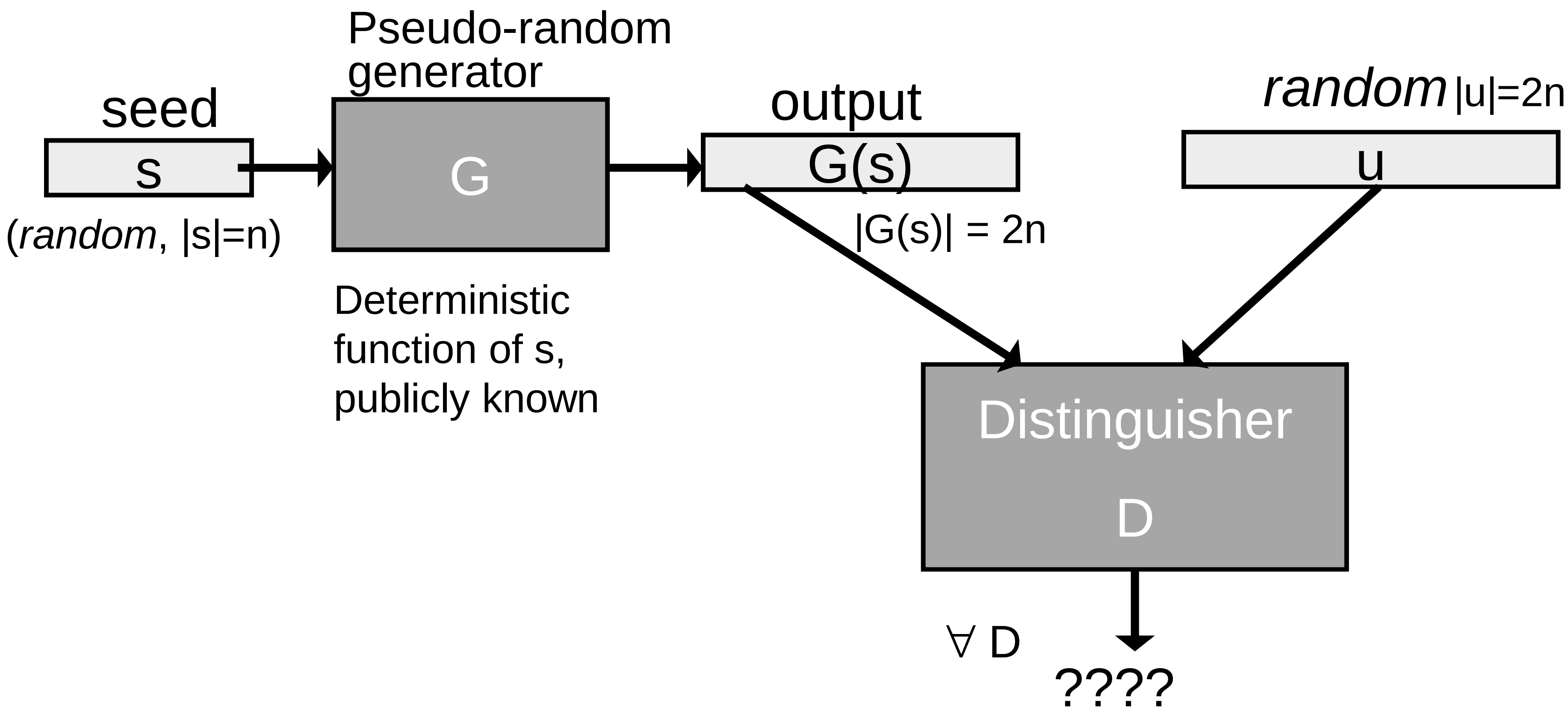
# Pseudo-random generator



# P vs. NP

- If P=NP then PRGs do not exist (why?)
- So their existence can only be conjectured until the P=NP question is resolved.

# Using a PRG for Encryption

- Replace the one-time-pad with the output of the PRG

- Key: a (short) random key $k \in \{0,1\}^{|k|}$.
- Message $m = m_1, \ldots, m_{|m|}$.
- Use a PRG $G : \{0,1\}^{|k|} \to \{0,1\}^{|m|}$
- Key generation: choose $k \in \{0,1\}^{|k|}$ uniformly at random.
- Encryption:
  - Use the output of the PRG as a one-time pad. Namely,
  - Generate $G(k) = g_1, \ldots, g_{|m|}$
  - Ciphertext $C = g_1 \oplus m_1, \ldots, g_{|m|} \oplus m_{|m|}$

- This is an example of a *stream cipher.*

# Security of encryption against polynomial adversaries

- Perfect security (previous equivalent defs):
  - (indistinguishability) $\forall$ $m_0,m_1 \in M$, $\forall c$, the probability that c is an encryption of $m_0$ is equal to the probability that c is an encryption of $m_1$.
  - (semantic security) The distribution of m given the encryption of m is the same as the a-priori distribution of m.
- Security of pseudo-random encryption (equivalent defs):
  - (indistinguishability) $\forall$ $m_0,m_1 \in M$, no *polynomial time* adversary D can distinguish between the encryptions of $m_0$ and of $m_1$. Namely, $Pr[D(E(m_0))=1] \approx Pr[D(E(m_1))=1)$
  - (semantic security) $\forall$ $m_0,m_1 \in M$, a polynomial time adversary which is given $E(m_b)$, where $b \in_r \{0,1\}$, succeeds in finding b with probability $\approx$ ½.

# RC4

- A stream cipher designed by Ron Rivest. Intellectual property belongs to RSA Inc.
  - Designed in 1987.
  - Kept secret until the design was leaked in 1994.

- Used in many protocols (SSL, etc.)

- Byte oriented operations.
- 8-16 machine operations per output byte.
- First output bytes are biased ☹

# RC4 initialization

Word size is a single byte.

Input: $k_0; \ldots; k_{255}$ (if key has fewer bits, pad it to itself sufficiently many times)

1. $j = 0$
2. $S_0 = 0;\ S_1 = 1; \ldots ;\ S_{255} = 255$
3. Let the key be $k_0; \ldots; k_{255}$
4. For $i = 0$ to $255$
   - $j = (j + S_i + k_i) \bmod 256$
   - Swap $S_i$ and $S_j$

(note that S is a permutation of 0,…,255)

# RC4 keying stream generation

An output byte B is generated as follows:

- $i = i + 1 \bmod 256$
- $j = j + S_i \bmod 256$
- Swap $S_i$ and $S_j$
- $r = S_i + S_j \bmod 256$
- Output: $B = S_r$

B is xored to the next byte of the plaintext.

(since S is a permutation, we want that B is uniformly distributed)

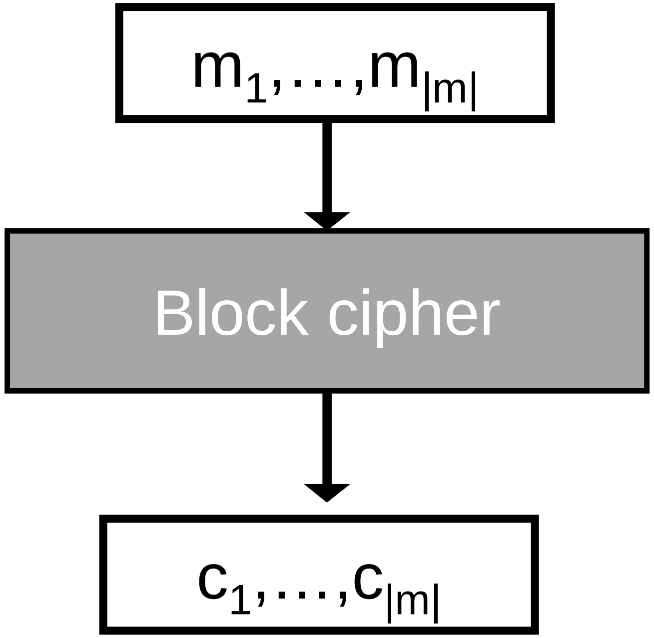
Bias: The probability that the first two output bytes are 0 is $2^{-16}+2^{-23}$

# Block Ciphers

- Plaintexts, ciphertexts of fixed length, |m|. Usually, |m|=64 or |m|=128 bits.
- The encryption algorithm $E_k$ is a *permutation* over $\{0,1\}^{|m|}$, and the decryption $D_k$ is its inverse. (They *are not* permutations of the bit order, but rather of the entire string.)

- Ideally, use a *random* permutation.
  - Can only be implemented using a table with $2^{|m|}$ entries ☹
- Instead, use a *pseudo-random* permutation*, keyed by a key k.
  - Implemented by a computer program whose input is m,k.

  - (*) will be explained shortly

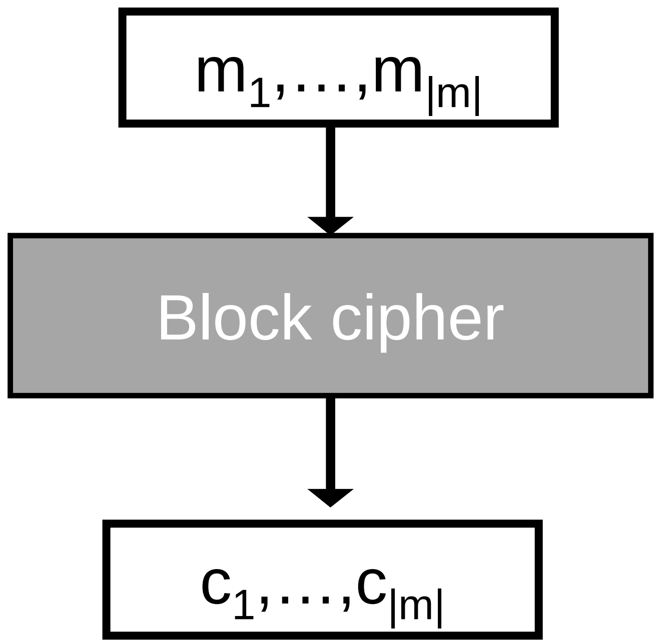


# Block Ciphers

- Modeled as a pseudo-random permutation.

- Encrypt/decrypt whole blocks of bits
  - Might provide better encryption by simultaneously working on a block of bits
  - One error in ciphertext affects whole block
  - Delay in encryption/decryption
  - There was more research on the security of block ciphers than on the security of stream ciphers.
  - Avoid the synchronization problem of stream cipher usage.

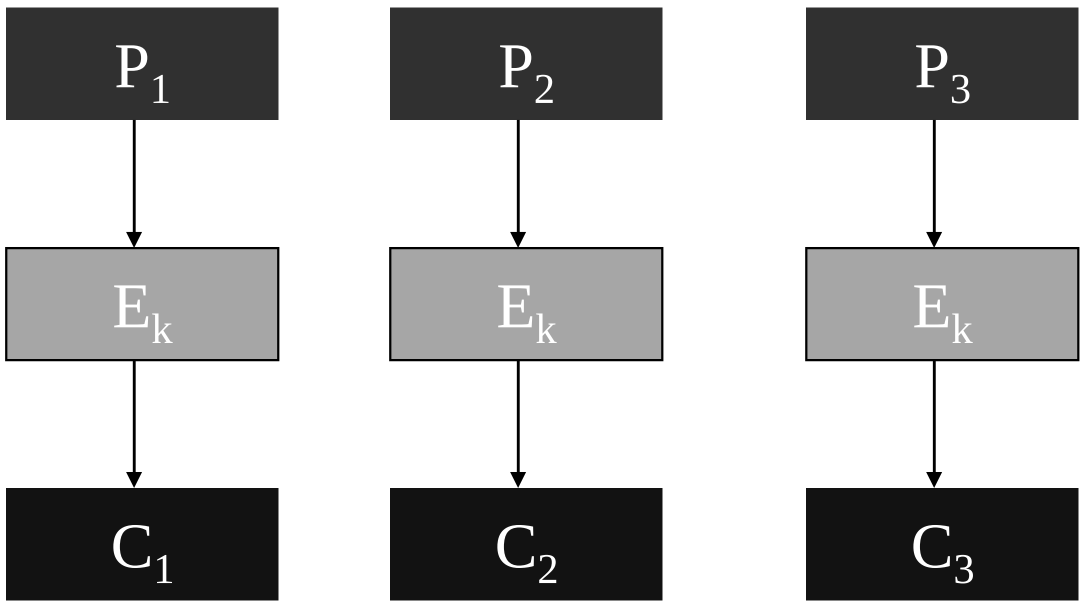
- Different *modes of operation* (for encrypting longer inputs)



# Block ciphers

- A block cipher is a function $F_k(x)$ of a key k and an |m| bit input x, which has an |m| bit output.
  - $F_k(x)$ is a keyed permutation
- How can we encrypt plaintexts longer than |m|?
- Different modes of operation were designed for this task.

# ECB Encryption Mode (Electronic Code Book)



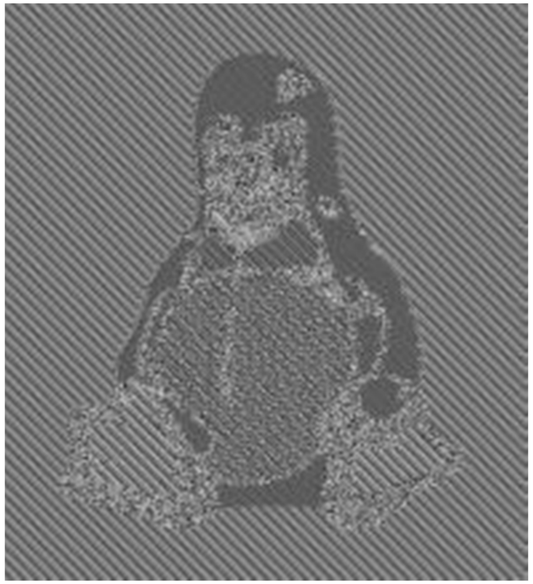

Namely, encrypt each plaintext block separately.

# Properties of ECB

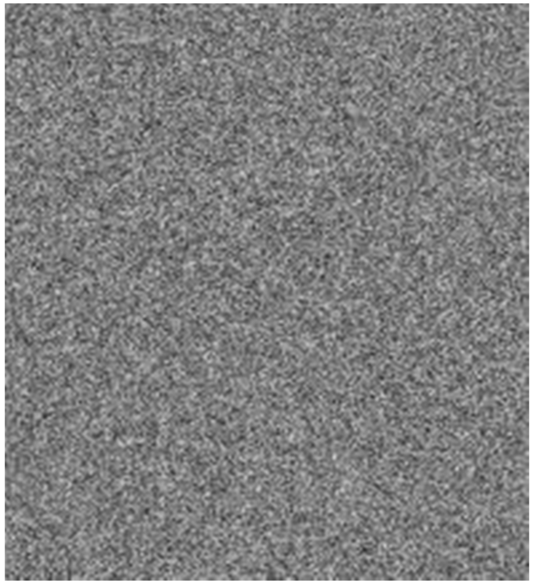
- Simple and efficient ☺
- Parallel implementation is possible ☺
- Does not conceal plaintext patterns ☹
  - Enc($P_1$, $P_2$, $P_1$, $P_3$)
- Active attacks are easy ☹ (plaintext can be easily manipulated by removing, repeating, or interchanging blocks).

# Encrypting bitmap images in ECB mode

original

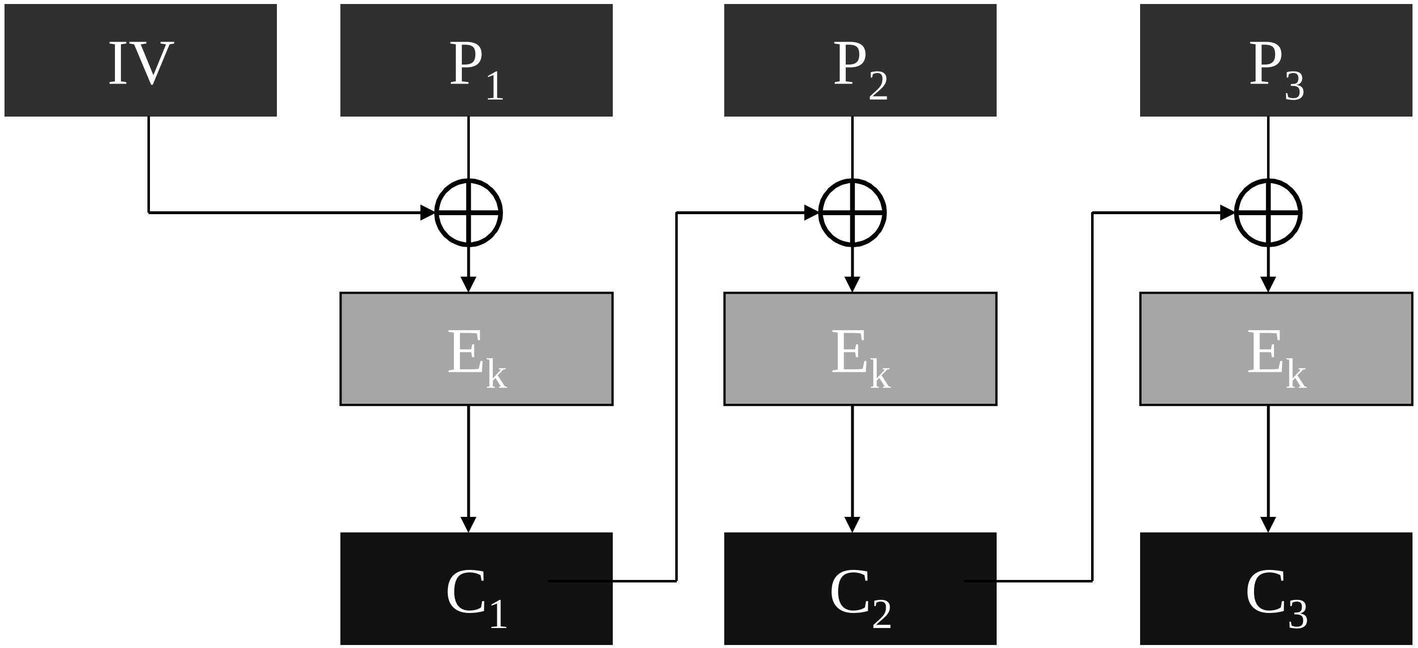


encrypted using
ECB mode



encrypted using
a secure mode

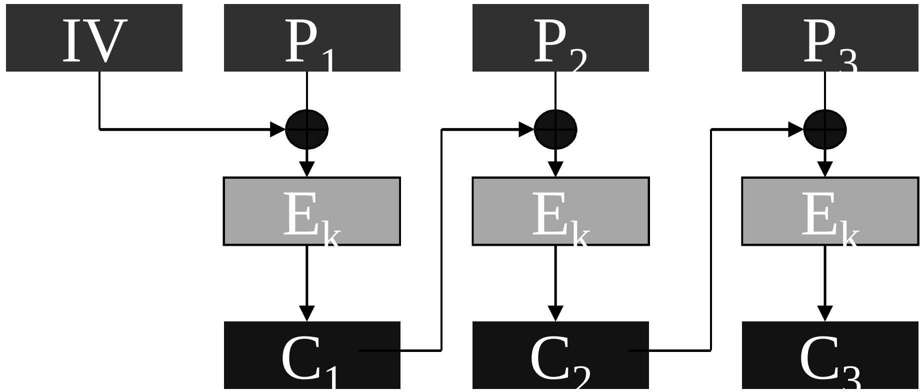
# CBC Encryption Mode (Cipher Block Chaining)



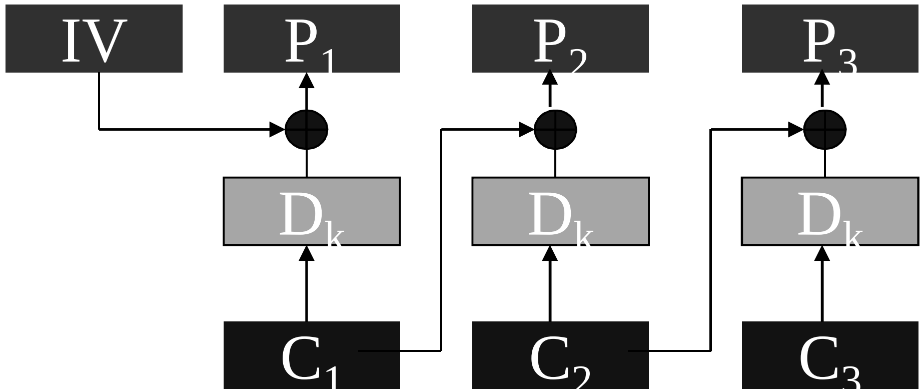

Previous *ciphertext* is XORed with current *plaintext* before encrypting current block.
An initialization vector IV is used as a “seed” for the process.
IV can be transmitted in the clear (unencrypted).

# CBC Mode

Encryption:



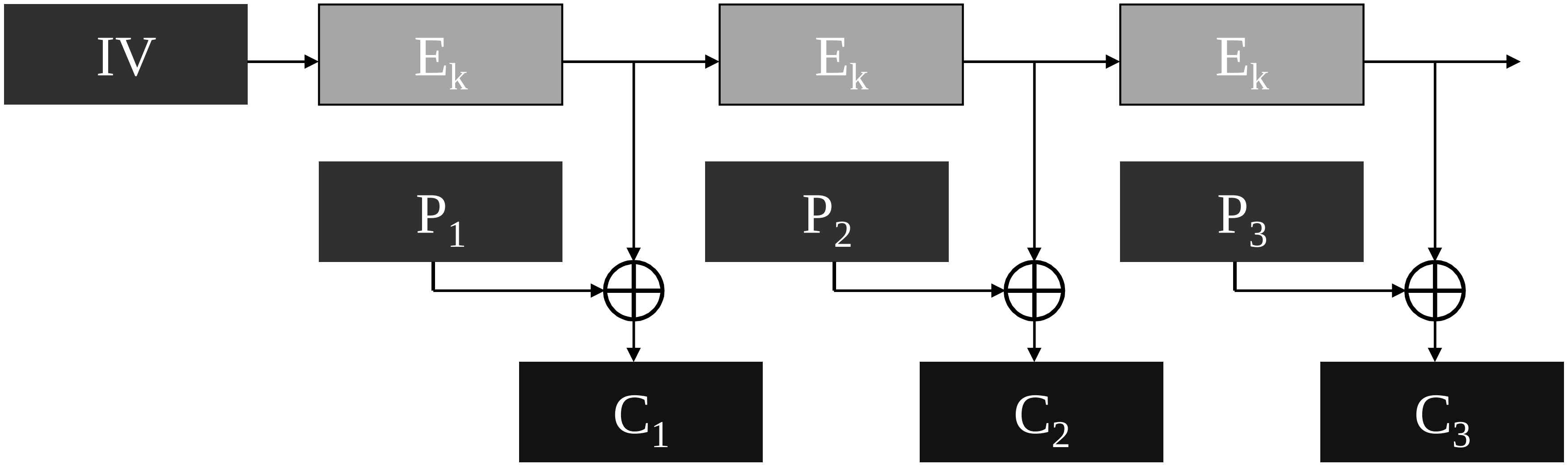

Decryption:



# Properties of CBC

- Asynchronous: the receiver can start decrypting from any block in the ciphertext. ☺
- Errors in one *ciphertext* block propagate to the decryption of the next block (but that's it). ☺
- Conceals plaintext patterns (same block ⇒ different ciphertext blocks) ☺
  - If IV is chosen at random, and $E_K$ is a pseudo-random permutation, CBC provides chosen-plaintext security.
  - But if IV is fixed, CBC does not even hide not common *prefixes.*
- No parallel implementation is known ☹
- Plaintext cannot be easily manipulated ☺
- Standard in most systems: SSL, IPSec, etc.

# OFB Mode (Output FeedBack)




- An initialization vector IV is used as a “seed” for generating a sequence of “pad” blocks
  - $E_k(IV), E_k(E_k(IV)), E_k(E_k(E_k(IV))),\ldots$
- Essentially a stream cipher.
- IV can be sent in the clear. Must never be repeated.

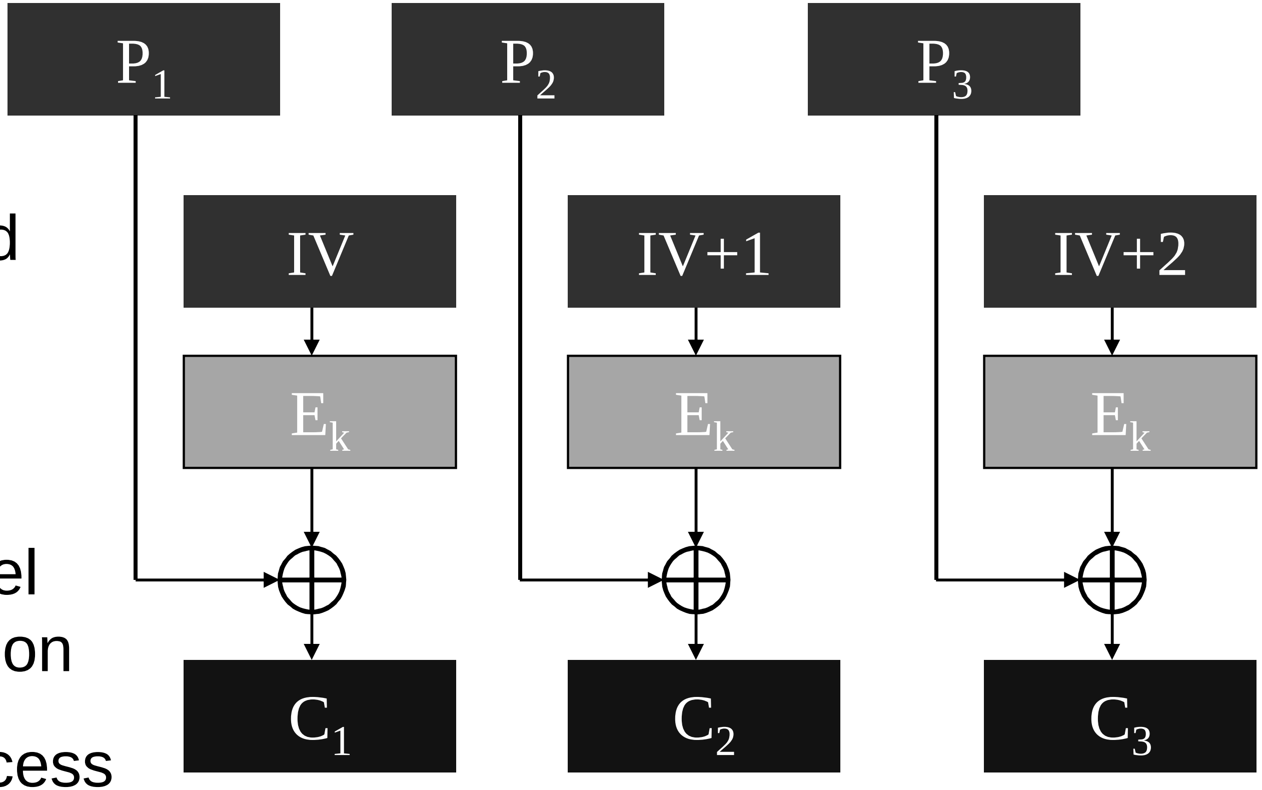
# Properties of OFB

- Essentially implements a synchronous stream cipher. I.e., the two parties must know $s_0$ and the current bit position.
  - A block cipher can be used instead of a PRG.
  - The parties must synchronize the location they are encrypting/decrypting. ☹

- Conceals plaintext patterns. If IV is chosen at random, and $E_K$ is a pseudo-random permutation, CBC provides chosen-plaintext security. ☺

- Errors in ciphertext do not propagate ☺
- Implementation:
  - Pre-processing is possible ☺
  - No parallel implementation is known ☹
- Active attacks (by manipulating the plaintext) are possible ☹

# CTR (counter) Encryption Mode




IV is selected as a random value

- easy parallel implementation
- <u>random access</u>
- preprocessing

# Pseudo-random functions

- A pseudo-random function is a function which cannot be distinguished from a random function.
  - The possible number of functions $f : \{0,1\}^n \rightarrow \{0,1\}^l$ is $2^{2^n l}$
  - A random function is one which is chosen at random from that range. Its representation must be at least $2^n l$ bits.
  - Alternatively, we can say that the random function chooses the value of $f(x)$ independently at random for every x.

# Pseudo-random functions - definition

- $F : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$
  - The first input is the key, and once chosen it is kept fixed.
  - For simplicity, assume $F : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^n$
  - $F(k,x)$ is written as $F_k(x)$

- F is pseudo-random if $F_k()$ (where k is chosen uniformly at random) is indistinguishable (to a polynomial distinguisher D) from a function *f* chosen at random from all functions mapping $\{0,1\}^n$ to $\{0,1\}^n$
  - There are $2^n$ choices of $F_k$, whereas there are $(2^n)^{2^n}$ choices for *f*.
  - The distinguisher D's task:
    - We choose a function G. With probability ½ G is $F_k$ (where $k \in_R \{0,1\}^n$), and with probability ½ it is a random function *f*.
    - D can compute $G(x_1),G(x_2),\ldots$ for any $x_1,x_2,\ldots$ it chooses.
    - D must output 1 if $G=F_k$.
    - $F_k$ is pseudo-random if $|\Pr(D(Fk)=1)-\Pr(D(G)=1)| \leq$ negligible.

# Pseudo-random permutations

- $F_k(x)$ is a keyed permutation if for every choice of k, $F_k()$ is one-to-one.
  - Note that in this case $F_k(x)$ has an inverse, namely for every y there is exactly one x for which $F_k(x)=y$.

- $F_k(x)$ is a pseudo-random permutation if
  - It is a keyed permutation
  - It is indistinguishable (to a polynomial distinguisher D) from a permutation *f* chosen at random from all permutations mapping $\{0,1\}^n$ to $\{0,1\}^n$.
    - $2^n$ possible values for $F_k$
    - $(2^n)!$ possible values for a random permutation

- Block ciphers are modeled as pseudo-random permutations.

- However, even a random permutation leaks some information if it is used to encrypt longer messages
  - Identical blocks result in identical ciphertexts.

- A stronger definition of security, and an appropriate construction are needed to prevent this information leakage.

# CPA security of block ciphers

- CPA (chosen-plaintext attack) indistinguishability
  - A key k is chosen at random
  - The adversary is given access to $E_k()$, and can encrypt any message it wants.
  - The adversary *A* chooses two messages $m_0$,$m_1$.
  - A random message $m_b$ is chosen, $b \in \{0,1\}$.
  - *A* is given a challenge ciphertext $E_k(m_b)$.
  - *A* can continue to compute $E_k()$ on any message.
  - *A* must output b'.
  - *A* succeeds if b=b'.
- The encryption scheme is (t,e)-CPA-secure if for all *A* that runs at most t steps, Pr(b=b') < 1/2+e.

# Constructing CPA-secure encryption

- Note that the encryption must be probabilistic.

- Let $F: \{0,1\}^n \rightarrow \{0,1\}^n$ be a pseudo-random function.
- The construction
  - Choose a random key $k \in \{0,1\}^n$
  - Encryption of $m \in \{0,1\}^n$: choose random $r \in \{0,1\}^n$, output $c = (r, F_k(r) \oplus m)$.
  - Decryption of $c = (r, f)$: compute $m = F_k(r) \oplus f$.

  - Intuitively, $F_k(r)$ is indistinguishable from a random message, and therefore ciphertext is like a one-time pad.

# Security

- Theorem: If $F_k$ is a pseudo-random function then the encryption scheme is $(t,\varepsilon)$-CPA-indistinguishable.
- Proof sketch:
  - If $F_k$ is random, then the adversary learns something only if the challenge ciphertext is $(r, F_k(r) \oplus m)$, and r was used in one of the encryptions asked by the adversary.
  - The prob. of this happening is $< t / 2^n$.
  - Replace the random function with a pseudo-random one.
    - Need to show that this change does not affect the probability of success in more than a negligible $\varepsilon$. (see next page)
  - Therefore total success probability is $< ½ + t/2^n + \varepsilon$.

# Security (contd.)

Background:

- If $F_k$ is random, then the adversary succeeds with prob < $t / 2^n$.

– Replace the random function with a pseudo-random $F_k$.

– Suppose that now success probability is > $½ + t/2^n + p(n)$.

– Then we found a distinguisher D between $F_k$ and a random function, which succeeds with prob > p(n).

- D has oracle access to a function G which is either random or is the prf $F_k$ , and to an attacker A against the encryption.
- D constructs an encryption according to the construction, and lets A attack it. Whenever A asks for an encryption, D asks for a value of G and encrypts.
- If A succeeds in decryption, D claims that G is the prf. Otherwise D claims that G is random. $|Pr(D(Fk)=1)-Pr(D(G)=1)| = p(n) > neg.$