Introduction to Cryptography

Lecture 9

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Hard problems in cyclic groups of prime order

- The following problems are believed to be hard in subgroups of prime order of Z_p^* (if the subgroup is large enough)
  - The discrete log problem
  - The Diffie-Hellman problem: The input contains g and $x,y \in G$, such that $x=g^a$ and $y=g^b$ (where a,b were chosen at random). The task is to find $z=g^{a \cdot b}$.
  - The Decisional Diffie-Hellman problem: The input contains $x,y \in G$, such that $x=g^a$ and $y=g^b$ (and a,b were chosen at random); and a pair (z,z') where one of (z,z') is $g^{a \cdot b}$ and the other is g^c (for a random c). The task is to tell which of (z,z') is $g^{a \cdot b}$.
- Solving DDH $\leq$ solving DH $\leq$ solving DL
  - All believed to be hard if the size of the subgroup > 2^{700}.

The Diffie-Hellman Key Exchange Protocol

• Public parameters: a group where the DDH assumption holds. For example, a subgroup $H \subset Z_p^*$ (where $|p|= 768$ or 1024, $p=2q+1$) of order q, and a generator g of $H \subset Z_p^*$.

- Alice:
  - picks a random $a \in [1,q]$.
  - Sends g^a mod p *to Bob.*
  - *Computes* $k=(g^b)^a$ mod p
- Bob:
  - picks a random $b \in [1,q]$.
  - Sends g^b mod p *to Bob.*
  - *Computes* $k=(g^a)^b$ mod p

• $K = g^{ab}$ is used as a shared key between Alice and Bob.

  • DDH assumption $\Rightarrow$ K is indistinguishable from a random key

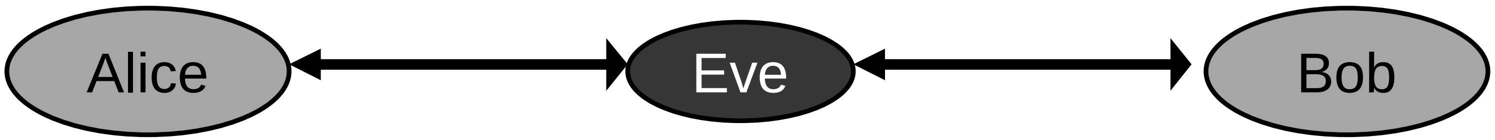
Diffie-Hellman: security

- A *(passive)* adversary
  - Knows Z_p^*, g
  - Sees g^a, g^b
  - Wants to compute g^{ab}, or at least learn something about it
- Recall the Decisional Diffie-Hellman problem:
  - Given random $x,y \in Z_p^*$, such that $x=g^a$ and $y=g^b$; *and a pair* (g^{ab},g^c) (in random order, for a random c), it is hard to tell which is g^{ab}.
  - An adversary that distinguishes the key g^{ab} generated in a DH key exchange from random, can also break the DDH.

  - *Note:* it is insufficient to require that the adversary cannot compute g^{ab}.

An active attack against the Diffie-Hellman Key Exchange Protocol

- An active adversary Eve.
- Can read and change the communication between Alice and Bob.
- …As if Alice and Bob communicate via Eve.



Man–in-the-Middle: an active attack against the Diffie-Hellman Key Exchange protocol

- Alice:
  - picks a random $a \in [1,q]$.
  - Sends g^a mod p *to Bob.*

Eve changes g^a to g^c

- Bob:
  - picks a random $b \in [1,q]$.
  - Sends g^b mod p to Alice.

Eve changes g^b to g^d

- *Computes* $k=(g^d)^a$ mod p
- *Computes* $k=(g^c)^b$ mod p

Keys:		
Alice	Eve	Bob
g^{ad}	g^{ad}, g^{bc}	g^{bc}

- Solution: ? (wireless usb)

Public key encryption

- Alice publishes a *public* key PK_{Alice}.
- Alice has a *secret* key SK_{Alice}.
- Anyone knowing PK_{Alice} can encrypt messages using it.
- Message decryption is possible only if SK_{Alice} is known.

- Compared to symmetric encryption:
  - Easier key management: n users need n keys, rather than $O(n^2)$ keys, to communicate securely.
- Compared to Diffie-Hellman key agreement:
  - No need for an interactive key agreement protocol. (Think about sending email…)

- Secure as long as we can trust the association of keys with users.

Public key encryption

- Must have different keys for encryption and decryption.

- Public key encryption cannot provide perfect secrecy:
  - Suppose $E_{pk}()$ is an algorithm that encrypts m=0/1, and uses r random bits in operation.
  - An adversary is given $E_{pk}(m)$. It can compare it to all possible 2^r encryptions of 0…

- Efficiency is the main drawback of public key encryption.

Defining a public key encryption

- The definition must include the following algorithms;
- Key generation: KeyGen(1^k)$\rightarrow$(PK,SK) (where k is a security parameter, e.g. k=1000).
- Encryption: C = E_{PK}(m) (E might be a randomized algorithm)
- Decryption: M= D_{SK}(C)

The El Gamal public key encryption system

- Public information (can be common to different public keys):
  - A group in which the DDH assumption holds. Usually start with a prime $p=2q+1$, and use $H \subset Z_p^*$ of order q. Define a generator g of H.

- Key generation: pick a random private key a in [1,|H|] (e.g. $0<a<q$). Define the public key $h=g^a$ ($h=g^a \bmod p$).

- Encryption of a message $m \in H \subset Z_p^*$
  - Pick a random $0 < r < q$.
  - The ciphertext is $(g^r, h^r \cdot m)$.

Using public key alone

- Decryption of (s,t)
  - Compute t / s^a ($m = h^r \cdot m / (g^r)^a$)

Using private key

El Gamal and Diffie-Hellman

- ElGamal encryption is similar to DH key exchange
  - DH key exchange: Adversary sees g^a, g^b. Cannot distinguish the key g^{ab} *from random.*
  - El Gamal:
    - A fixed public key g^a.
    - Sender picks a random g^r.

    Known to the adversary

    - Sender encrypts message using g^{ar}.

    Used as a key

- El Gamal is like DH where
  - The same g^a is used for all communication
  - There is no need to explicitly send this g^a (it is already known as the public key of Alice)

The El Gamal public key encryption system

- Setting the public information
- *A large prime p, and a generator g of* $H \subset Z_p^*$ of order q.
  - $|p|$ = 756 or 1024 bits.
  - $p-1$ must have a large prime factor (e.g. $p=2q+1$)
    - Otherwise it is easy to solve discrete logs in Z_p^* (relevant also to DH key agreement)
    - This large prime factor is also needed for the DDH assumption to hold (Legendre's symbol).
  - g must be a generator of a large subgroup of Z_p^*, in which the DDH assumption holds.

The El Gamal public key encryption system

- Encoding the message:
  - m must be in the subgroup H generated by g.
  - If $p=2q+1$, and H is the subgroup of quadratic residues (which has $(p-1)/2=q$ items), we can map each message $m \in \{1,\ldots,(p-1)/2\}$ to the value $m^2 \bmod p$, which is in H.
    - Encrypt m^2 instead of m. Therefore decryption yields m^2 and not m^2. Must then compute a square root.
  - Alternatively, encrypt m using *$(g^r, H(h^r) \oplus m)$. Decryption is done by computing $H((g^r)^a)$.* *(H* is a hash function that preserves the pseudo-randomness of h^r.)

The El Gamal public key encryption system

- Overhead:
  - Encryption: two exponentiations; preprocessing possible.
  - Decryption: one exponentiation.
  - message expansion: $m \Rightarrow (g^r, h^r \cdot m)$.

- Randomized encryption
  - Must use fresh randomness *r* for every message.
  - Two different encryptions of the same message are different! (provides semantic security)

Security reductions

- Security by reduction
  - Define what it means for the system to be “secure” (chosen plaintext/ciphertext attacks, etc.)
  - State a “hardness assumption” (e.g., that it is hard to extract discrete logarithms in a certain group).
  - Show that if the hardness assumption holds then the cryptosystem is secure.
  - Usually prove security by showing that breaking the cryptosystem means that the hardness assumption is false.

- Benefits:
  - To examine the security of the system it is sufficient to check whether the assumption holds
  - Similarly, for setting parameters (e.g. group size).

Semantic security

- Semantic Security: knowing that an encryption is either $E(m_0)$ or $E(m_1)$, (where m_0,m_1 are known, or even chosen by the attacker) an adversary cannot decide with probability better than ½ which is the case.
  - This is a very strong security property.

- Suppose that a public key encryption system is deterministic., then it cannot have semantic security.
  - In this case, E(m) is a deterministic function of m and P.
  - Therefore, if Eve suspects that Bob might encrypt either m_0 or m_1, she can compute (by herself) $E(m_0)$ and $E(m_1)$ and compare them to the encryption that Bob sends.

Goal and method

- Goal
  - Show that if the DDH assumption holds
  - Then the El Gamal cryptosystem is semantically secure

- Method:
  - Show that if the El Gamal cryptosystem is *not* semantically secure
  - Then the DDH assumption *does not* hold

El Gamal encryption: breaking semantic security implies breaking DDH

- Proof by reduction:
  - We can use an adversay that breaks El Gamal.
  - We are given a DDH challenge: $(g,g^a,g^r, (D_0,D_1)\ )$ where one of D_0,D_1 is g^{ar}, and the other is g^c. We need to identify g^{ar}.
  - We give the adversay g and a public key: $h=g^a$.
  - The adversary chooses m_0,m_1.
  - We give the adversay $(g^r,D_e \cdot m_b)$, using random $b,e \in \{0,1\}$. (That is, choose m_b randomly from $\{m_0,m_1\}$, choose D_e randomly from $\{D_0,D_1\}$. The result is a valid El Gamal encryption if $D_e=g^{ar}$.)
  - If the adversay guesses b correctly, we decide that $D_e=g^{ar}$. Otherwise we decide that $D_e=g^c$.

El Gamal encryption: breaking semantic security implies breaking DDH

- Analysis:
  - Suppose that the adversary can break the El Gamal encryption with prob 1.
  - If $D_e=g^{ar}$ then the adversary finds c with probability 1, otherwise it finds c with probability ½.
  - Our success probability $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = 3/4$.

  - Suppose now that the adversary can break the El Gamal encryption with prob ½+p.
  - If $D_e=g^{ar}$ then the adversary finds c with probability ½+p, otherwise it finds c with probability ½.
  - Our success probability $\frac{1}{2} \cdot (\frac{1}{2}+p) + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}+\frac{1}{2}p$. QED

Chosen ciphertext attacks

- In a chosen ciphertext attack, the adversary is allowed to obtain decryptions of arbitrary ciphertexts of its choice (except for the specific message it needs to decrypt).
- El Gamal encryption is insecure against chosen ciphertext attacks:
  - Suppose the adversary wants to decrypt $<c_1,c_2>$ which is an ElGamal encryption of the form $(g^r,h^r m)$.
  - The adversary computes $c'_1=c_1 g^{r'}$, $c'_2=c_2 h^{r'} m'$, where it chooses r',m' at random.
  - It asks for the decryption of $<c'_1,c'_2>$. It multiplies the plaintext by $(m')^{-1}$ and obtains m.

Homomorphic property

- The attack on chosen ciphertext security is based on the homomorphic property of the encryption

- Homomorphic property:
  - Given encryptions of *x,y*, it is easy to generate an encryption of $x \cdot y$
    - $(g^r, h^r \cdot x) \times (g^{r'}, h^{r'} \cdot y) \rightarrow (g^{r''}, h^{r''} \cdot x \cdot y)$

Homomorphic encryption

- Homomorphic encryption is useful for performing operations over encrypted data.
- Given $E(m_1)$ and $E(m_2)$ it is easy to compute $E(m_1m_2)$, even if you don't know how to decrypt.

- For example, an election procedure:
  - A "Yes" is E(2). A "No" vote is E(1).
  - Take all the votes and multiply them. Obtain $E(2^j)$, where j is the number of "Yes" votes.
  - Decrypt only the result and find out how many "Yes" votes there are, without identifying how each person voted.

Integer Multiplication & Factoring as a One Way Function.

$$\begin{tikzcd}
p,q \arrow[rr, bend left, "\textbf{easy}"] & & N=pq \arrow[ll, bend left, "\textbf{hard}"]
\end{tikzcd}$$

Can a public key system be based on this observation ?????

Excerpts from RSA paper (CACM, 1978)

The era of "electronic mail" may soon be upon us; we must ensure that two important properties of the current "paper mail" system are preserved: (a) messages are *private*, and (b) messages can be *signed*. We demonstrate in this paper how to build these capabilities into an electronic mail system.

At the heart of our proposal is a new encryption method. This method provides an implementation of a "public-key cryptosystem," an elegant concept invented by Diffie and Hellman. Their article motivated our research, since they presented the concept but not any practical implementation of such system.