

Introduction to Cryptography

Lecture 7

Public key cryptography

Benny Pinkas

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Last lecture

- Basic number theory
 - Lots of facts about groups
- In particular
 - Z_p^* Multiplication modulo a prime number p
 - $(G, \circ) = (\{1, 2, \dots, p-1\}, \times)$, e.g., $Z_7^* = (\{1, 2, 3, 4, 5, 6\}, \times)$.
 - Z_N^* Multiplication modulo a composite number N
 - $(G, \circ) = (\{a \text{ s.t. } 1 \leq a \leq N-1 \text{ and } \gcd(a, N)=1\}, \times)$
 - E.g., $Z_{10}^* = (\{1, 3, 7, 9\}, \times)$
 - A group G is cyclic if there exists a generator g , s.t. $\forall a \in G, \exists i \text{ s.t. } g^i = a$.

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The Diffie-Hellman Key Exchange Protocol

- Public parameters: a group where the DDH assumption holds. For example, Z_p^* (where $|p| = 768$ or 1024 , $p=2q+1$), and a generator g of $H \subset Z_p^*$ of order q .
- Alice:
 - picks a random $a \in [1, q]$.
 - Sends $g^a \bmod p$ to Bob.
 - Computes $k = (g^b)^a \bmod p$
- Bob:
 - picks a random $b \in [1, q]$.
 - Sends $g^b \bmod p$ to Alice.
 - Computes $k = (g^a)^b \bmod p$
- $K = g^{ab}$ is used as a shared key between Alice and Bob.
 - DDH assumption $\Rightarrow K$ is indistinguishable from a random key

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Diffie-Hellman: security

- A (passive) adversary
 - Knows Z_p^*, g
 - Sees g^a, g^b
 - Wants to compute g^{ab} , or at least learn something about it
- Recall the Decisional Diffie-Hellman problem:
 - Given random $x, y \in Z_p^*$, such that $x = g^a$ and $y = g^b$; and a pair (g^{ab}, g^c) (in random order, for a random c), it is hard to tell which is g^{ab} .
 - An adversary that distinguishes the key g^{ab} generated in a DH key exchange from random, can also break the DDH.
 - Note: it is insufficient to require that the adversary cannot compute g^{ab} .

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Diffie-Hellman key exchange: usage

- The DH key exchange can be used in any group in which the Decisional Diffie-Hellman (DDH) assumption is believed to hold.
- Currently, Z_p^* and elliptic curve groups.
- Common usage:
 - Overhead: 1-2 exponentiations
 - Usually,
 - A DH key exchange for generating a master key
 - Master key used to encrypt session keys
 - Session key is used to encrypt traffic with a symmetric cryptosystem

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An active attack against the Diffie-Hellman Key Exchange Protocol

- An active adversary Eve.
- Can read and change the communication between Alice and Bob.
- ...As if Alice and Bob communicate via Eve.



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Man-in-the-Middle: an active attack against the Diffie-Hellman Key Exchange protocol

- Alice:
 - picks a random $a \in [1, q]$.
 - Sends $g^a \bmod p$ to Bob.
 - Bob:
 - picks a random $b \in [1, q]$.
 - Sends $g^b \bmod p$ to Alice.
 - Eve changes g^a to g^c
 - Eve changes g^b to g^d
 - Computes $k = (g^d)^a \bmod p$
 - Computes $k = (g^c)^b \bmod p$
- | Keys: | Alice | Eve | Bob |
|-------|----------|------------------|----------|
| | g^{ad} | g^{ad}, g^{bc} | g^{bc} |
- Solution: ? (wireless usb)

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Public key encryption

- Alice publishes a *public* key PK_{Alice} .
- Alice has a *secret* key SK_{Alice} .
- Anyone knowing PK_{Alice} can encrypt messages using it.
- Message decryption is possible only if SK_{Alice} is known.
- Compared to symmetric encryption:
 - Easier key management: n users need n keys, rather than $O(n^2)$ keys, to communicate securely.
- Compared to Diffie-Hellman key agreement:
 - No need for an interactive key agreement protocol. (Think about sending email...)
- Secure as long as we can trust the association of keys with users.

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Public key encryption

- Must have different keys for encryption and decryption.
- Public key encryption cannot provide perfect secrecy:
 - Suppose $E_{pk}()$ is an algorithm that encrypts $m=0/1$, and uses r random bits in operation.
 - An adversary is given $E_{pk}(m)$. It can compare it to all possible 2^r encryptions of 0...
- Efficiency is the main drawback of public key encryption.

Defining a public key encryption

- The definition must include the following algorithms;
- Key generation: $\text{KeyGen}(1^k) \rightarrow (PK, SK)$ (where k is a security parameter, e.g. $k=1000$).
- Encryption: $C = E_{PK}(m)$ (E might be a randomized algorithm)
- Decryption: $M = D_{SK}(C)$

The El Gamal public key encryption system

- Public information (can be common to different public keys):
 - A group in which the DDH assumption holds. Usually start with a prime $p=2q+1$, and use $H \subset \mathbb{Z}_p^*$ of order q . Define a generator g of H .
- Key generation: pick a random private key a in $[1, |H|]$ (e.g. $0 < a < q$). Define the public key $h = g^a$ ($h = g^a \bmod p$).
- Encryption of a message $m \in H \subset \mathbb{Z}_p^*$
 - Pick a random $0 < r < q$.
 - The ciphertext is $(g^r, h^r \cdot m)$.

} Using public key alone
- Decryption of (s, t)
 - Compute t / s^a ($m = h^r \cdot m / (g^r)^a$)

} Using private key

El Gamal and Diffie-Hellman

- ElGamal encryption is similar to DH key exchange
 - DH key exchange: Adversary sees g^a, g^b . Cannot distinguish the key g^{ab} from random.
 - El Gamal:
 - A fixed public key g^a .
 - Sender picks a random g^r .
 - Sender encrypts message using g^{ar} .

} Known to the adversary
} Used as a key
- El Gamal is like DH where
 - The same g^a is used for all communication
 - There is no need to explicitly send this g^a (it is already known as the public key of Alice)

Semantic security

- Semantic Security: knowing that an encryption is either $E(m_0)$ or $E(m_1)$, (where m_0, m_1 are known) an adversary cannot decide with probability better than $\frac{1}{2}$ which is the case.
- Suppose that a public key encryption system is deterministic., then it cannot have semantic security.
 - Namely, $E(m)$ is a deterministic function of m and P .
 - Then if Eve suspects that Bob might encrypt either m_0 or m_1 , she can compute (by herself) $E(m_0)$ and $E(m_1)$ and compare them to the encryption that Bob sends.

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El Gamal encryption: breaking semantic security implies breaking DDH

- Proof by reduction:
 - We are given $(g, g^a, g^b, (D_1, D_2))$ where one of D_1, D_2 is g^{ab} , and the other is g^r . We need to identify g^{ab} .
 - We give the adversary g and a public key: $h = g^a$.
 - The adversary chooses m_0, m_1 .
 - We give the adversary $(g^b, D_e \cdot m_c)$, where c, e are random.
 - If the adversary guesses c correctly, we decide that $D_e = g^{ab}$. Otherwise we decide that $D_e = g^r$.
- Analysis:
 - Suppose that the adversary can guess c with prob $\frac{3}{4}$.
 - If $D_e = g^{ab}$ then the adversary finds c with probability $\frac{3}{4}$, otherwise it finds c with probability $\frac{1}{2}$.
 - Our success probability $\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} = 5/8$.

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The El Gamal public key encryption system

- Setting the public information
- A large prime p , and a generator g of $H \subset \mathbb{Z}_p^*$ of order q .
 - $|p| = 756$ or 1024 bits.
 - $p-1$ must have a large prime factor (e.g. $p=2q+1$)
 - Otherwise it is easy to solve discrete logs in \mathbb{Z}_p^* (relevant also to DH key agreement)
 - Needed for the DDH assumption to hold (Legendre's symbol)
 - g must be a generator of a large subgroup of \mathbb{Z}_p^* .
- Encoding the message:
 - m must be in the subgroup generated by g .
 - Alternatively, encrypt m using $(g^r, H(h^r) \oplus m)$. Decryption is done by computing $H((g^r)^a)$. (H is a hash function that preserves the pseudo-randomness of h^r .)

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The El Gamal public key encryption system

- Overhead:
 - Encryption: two exponentiations; preprocessing possible.
 - Decryption: one exponentiation.
 - message expansion: $m \Rightarrow (g^r, h^r \cdot m)$.
- Randomized encryption
 - Must use fresh randomness r for every message.
 - Two different encryptions of the same message are different! (provides semantic security)

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Homomorphic property

- Insecurity against chosen ciphertext attacks:
 - Attacker wants to decrypt $(s, t) = (g^r, h^r \cdot m)$.
 - Chooses random r' , computes $(s', t') = (s, t \cdot r') = (g^r, h^r \cdot (m \cdot r'))$.
 - Asks for a decryption of (s', t') . Receives $m \cdot r'$.
- Homomorphic property:
 - Given encryptions of x, y , it's easy to generate an encryption of $x \cdot y$.
 - $(g^r, h^r \cdot x) \times (g^{r'}, h^{r'} \cdot y) \rightarrow (g^{r+r'}, h^{r+r'} \cdot x \cdot y)$

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Homomorphic encryption

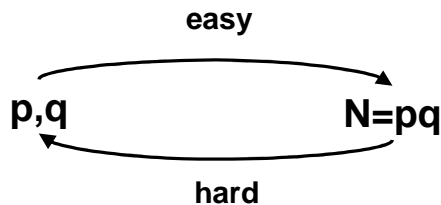
- Homomorphic encryption is useful for performing operations over encrypted data.
- Given $E(m_1)$ and $E(m_2)$ it is easy to compute $E(m_1 m_2)$.
- For example, an election procedure:
 - A “Yes” is $E(2)$. A “No” vote is $E(1)$.
 - Take all the votes and multiply them. Obtain $E(2^j)$, where j is the number of “Yes” votes.
 - Decrypt the result and find out how many “Yes” votes there are, without identifying how each person voted.

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Integer Multiplication & Factoring as a One Way Function.



Can a public key system be based on this observation ?????

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Excerpts from RSA paper (CACM, 1978)

The era of “electronic mail” may soon be upon us; we must ensure that two important properties of the current “paper mail” system are preserved: (a) messages are *private*, and (b) messages can be *signed*. We demonstrate in this paper how to build these capabilities into an electronic mail system.

At the heart of our proposal is a new encryption method. This method provides an implementation of a “public-key cryptosystem,” an elegant concept invented by Diffie and Hellman. Their article motivated our research, since they presented the concept but not any practical implementation of such system.

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The Multiplicative Group Z_{pq}^*

- p and q denote two large primes (e.g. 512 bits long).
- Denote their product as $N = pq$.
- The multiplicative group $Z_N^* = Z_{pq}^*$ contains all integers in the range $[1, pq-1]$ that are relatively prime to both p and q .
- The size of the group is
 - $\phi(n) = \phi(pq) = (p-1)(q-1) = N - (p+q) + 1$
- For every $x \in Z_N^*$, $x^{\phi(N)} = x^{(p-1)(q-1)} = 1 \pmod N$.

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Exponentiation in Z_N^*

- Motivation: use exponentiation for encryption.
- Let e be an integer, $1 < e < \phi(N) = (p-1)(q-1)$.
 - Question: When is exponentiation to the e^{th} power, $(x \rightarrow x^e)$, a one-to-one operation in Z_N^* ?
- Claim: If e is relatively prime to $(p-1)(q-1)$ then $x \rightarrow x^e$ is a one-to-one operation in Z_N^* .
- Constructive proof:
 - Since $\gcd(e, (p-1)(q-1)) = 1$, e has a multiplicative inverse modulo $(p-1)(q-1)$.
 - Denote it by d , then $ed = 1 + c(p-1)(q-1) = 1 + c\phi(N)$.
 - Let $y = x^e$, then $y^d = (x^e)^d = x^{1+c\phi(N)} = x$.
 - I.e., $y \rightarrow y^d$ is the inverse of $x \rightarrow x^e$.

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The RSA Public Key Cryptosystem

- Public key:
 - $N=pq$ the product of two primes (we assume that factoring N is hard)
 - e such that $\gcd(e, \phi(N)) = 1$ (are these hard to find?)
- Private key:
 - d such that $de \equiv 1 \pmod{\phi(N)}$
- Encryption of $M \in Z_N^*$
 - $C = E(M) = M^e \pmod N$
- Decryption of $C \in Z_N^*$
 - $M = D(C) = C^d \pmod N$ (why does it work?)

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Constructing an instance of the RSA PKC

- Alice
 - picks at random two large primes, p and q .
 - picks (uniformly at random) a (large) d that is relatively prime to $(p-1)(q-1)$ (namely, $\gcd(d, \phi(N)) = 1$).
 - Alice computes e such that $de \equiv 1 \pmod{\phi(N)}$
- Let $N=pq$ be the product of p and q .
- Alice publishes the public key (N, e) .
- Alice keeps the private key d , as well as the primes p, q and the number $\phi(N)$, in a safe place.

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Properties of RSA

- Deterministic encryption. In textbook RSA:
 - M is always encrypted as M^e
 - The ciphertext is as long as the domain of M
- Corollary: RSA does not have semantic security.
- Chosen ciphertext attack: (homomorphic property)
 - RSA is susceptible to chosen ciphertext attacks:
 - Given a ciphertext $C=M^e$, choose a random R and generate $C'=CR^e$ (an encryption of $M \cdot R$). Decrypting C' reveals M .

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Efficiency

- The public exponent e may be small.
 - It is common to choose its value to be either 3 or $2^{16}+1$. The private key d must be long.
 - Each encryption involves only a few modular multiplications. Decryption requires a full exponentiation.
- Usage of a small $e \Rightarrow$ Encryption is more efficient than a full blown exponentiation.
- Decryption requires a full exponentiation ($M=C^d \bmod N$)
- Can this be improved?

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The Chinese Remainder Theorem (CRT)

- Thm:
 - Let $N=pq$ with $\gcd(p,q)=1$.
 - Then for every pair $(y,z) \in \mathbb{Z}_p \times \mathbb{Z}_q$ there exists a *unique* $x \in \mathbb{Z}_N$, s.t.
 - $x=y \bmod p$
 - $x=z \bmod q$
- Proof:
 - The extended Euclidean algorithm finds a,b s.t. $ap+bq=1$.
 - Define $c=bq$. $c=1 \bmod p$. $c=0 \bmod q$.
 - Define $d=ap$. $d=0 \bmod p$. $d=1 \bmod q$.
 - Let $x=cy+dz \bmod N$.
 - $cy+dz = 1y + 0 = y \bmod p$.
 - $cy+dz = 0 + 1z = z \bmod q$.
 - (How efficient is this?)
 - (The inverse operation, finding (y,z) from x , is easy.)

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More efficient RSA decryption

- CRT:
 - Given p,q compute a,b s.t. $ap+bq=1$.
 - $c=bq$; $d=ap$
- } Once for all messages
- Decryption, given C :
 - Compute $y'=C^d \bmod p$. (instead of d can use $d'=d \bmod p-1$)
 - Compute $z'=C^d \bmod q$. (instead of d can use $d''=d \bmod q-1$)
 - Compute $M=cy'+dz' \bmod N$.
 - Overhead:
 - Two exponentiations modulo p,q , instead of one exponentiation modulo N .
 - Overhead of exponentiation is cubic in length of modulus.
 - i.e., save a factor of $2^{3/2}$.

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Security reductions

- Security by reduction
 - Define what it means for the system to be “secure” (chosen plaintext/ciphertext attacks, etc.)
 - State a “hardness assumption” (e.g., that it is hard to extract discrete logarithms in a certain group).
 - Show that if the hardness assumption holds then the cryptosystem is secure.
- Benefits:
 - To examine the security of the system it is sufficient to check whether the assumption holds
 - Similarly, for setting parameters (e.g. group size).

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RSA Security

- If factoring N is easy then RSA is insecure
 - (factor $N \Rightarrow$ find $p, q \Rightarrow$ find $(p-1)(q-1) \Rightarrow$ find d from e)
- Factoring assumption:
 - For a randomly chosen p, q of appropriate length, it is infeasible to factor $N=pq$.
- This assumption might be too weak (might not ensure secure encryption)
 - Maybe it's possible to break RSA without factoring N ?
 - We don't know how to reduce RSA security to the hardness of factoring.
- Fact: finding d is equivalent to factoring.
 - I.e., if it is possible to find d given (N, e) , then it is easy to factor N .
- “hardness of finding d assumption” no stronger than hardness of factoring.

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The RSA assumption: Trap-Door One-Way Function (OWF)

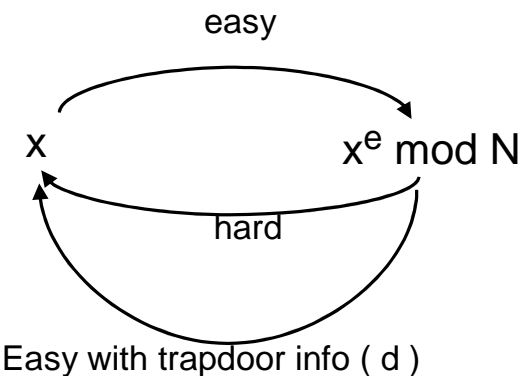
- (what is the minimal assumption required to show that RSA encryption is secure?)
- (Informal) definition: $f : D \rightarrow R$ is a *trapdoor one way function* if there is a trap-door s such that:
 - Without knowledge of s , the function f is a one way. I.e., for a randomly chosen x , it is hard to invert $f(x)$.
 - Given s , inverting f is easy
- Example: $f_{g,p}(x) = g^x \bmod p$ is *not* a trapdoor one way function.
- Example: assuming that RSA is a trapdoor OWF
 - $f_{N,e}(x) = x^e \bmod N$. (assumption: for a random N, e, x , inverting is hard.)
 - The trapdoor is d s.t. $ed = 1 \bmod \phi(N)$
 - $[f_{N,e}(x)]^d = x \bmod N$

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RSA as a One Way Trapdoor Permutation



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RSA assumption: cautions

- The RSA assumption is quite well established:
 - RSA is a Trapdoor One-Way Permutation
 - Hard to invert on random input – without secret key
- But is it a secure cryptosystem?
 - Given the assumption it is hard to reconstruct the input, but is it hard to learn *anything* about the input?
- Theorem [G]: RSA hides the $\log(\log(n))$ least *and* most significant bits of a uniformly-distributed random input
 - But some (other) information about pre-image may leak
 - And... adversary can detect a repeating message

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Is it safe to use a common modulus ?

- Consider the following environment:
 - There is a global modulus N . No one knows its factoring.
 - Each party has a pair (e_i, d_i) , such that $e_i d_i = 1 \bmod N$.
 - Used as a public/private key pair.
- The system is insecure.
- Party 1, knowing (e_1, d_1)
 - can factor N
 - Find d_i for any other party i .

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RSA with a small exponent

- Setting $e=3$ enables efficient encryption
- Might be insecure if not used properly
 - Assume three users with public keys N_1, N_2, N_3 .
 - Alice encrypts the same message to all of them
 - $C_1 = m^3 \bmod N_1$
 - $C_2 = m^3 \bmod N_2$
 - $C_3 = m^3 \bmod N_3$
- Can an adversary which sees C_1, C_2, C_3 find m ?
 - $m^3 < N_1 N_2 N_3$
 - N_1, N_2 and N_3 are most likely relatively prime (otherwise can factor).
 - Chinese remainder theorem -> can find $m^3 \bmod N$ (and therefore m^3 over the integers)
 - Easy to extract 3rd root over the integers.

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