Introduction to Cryptography
Lecture 6

Basic Number Theory,
Diffie-Hellman Key Exchange

Benny Pinkas

Last lecture

- Basic number theory
  - Integer numbers, divisors, primes
  - Modular operations
  - gcd algorithm
  - Extended gcd algorithm
    - Given a,b finds s,t such that gcd(a,b) = a·s + b·t
    - There is no common divisor smaller than gcd(a,b) which can be represented as a linear combination of a,b
      - For example, a=12, b=8.
      - 4= 1·12 - 1·8
      - There are no s,t for which 2=s·12 + t·8

Groups

- Definition: a set G with a binary operation $\circ$:G$\times$G$\rightarrow$G is called a group if:
  - (closure) $\forall a,b \in G$, it holds that $a \circ b \in G.$
  - (associativity) $\forall a,b,c \in G$, $(a \circ b) \circ c = a \circ (b \circ c).$
  - (identity element) $\exists e \in G,$ s.t. $\forall a \in G$ it holds that $a \circ e = a.$
  - (inverse element) $\forall a \in G \; \exists a^{-1} \in G,$ s.t. a $\circ a^{-1} = e.$
- A group is Abelian (commutative) if $\forall a,b \in G$, it holds that $a \circ b = b \circ a.$

- Examples:
  - Integers under addition
    - (Z,+) = {…,-3,-2,-1,0,1,2,3,…}

More examples of groups

- Addition modulo N
  - $(G, \circ) = (\{0,1,2,\dots,N-1\}, +)$

- Z_p^* Multiplication modulo a prime number p
  - $(G, \circ) = (\{1,2,\dots,p-1\}, \times)$
  - E.g., $Z_7^* = (\{1,2,3,4,5,6\}, \times)$

- Trivial: closure (the result of the multiplication is never divisible by *p*), associativity, existence of identity element.
- The extended GCD algorithm shows that an inverse always exists:
  - $s \cdot a + t \cdot p = 1 \Rightarrow s \cdot a = 1 - t \cdot p \Rightarrow s \cdot a \equiv 1 \bmod p$

More examples of groups

- Z_N^* Multiplication modulo a composite number N
  - $(G, \circ) = (\{a \text{ s.t. } 1 \leq a \leq N\text{-}1$ and $gcd(a,N)=1\}, \times)$
  - *E.g.*, $Z_{10}^* = (\{1,3,7,9\}, \times)$

  - Closure:
    - $s \cdot a+t \cdot N = 1$
    - $s' \cdot b+t' \cdot N = 1$
    - $ss' \cdot (ab)+(sat'+s'bt+ tt'N) \cdot N = 1$
    - Therefore $1=gcd(ab,N)$.
  - Associativity: trivial
  - Existence of identity element: 1.
  - Inverse element: as in Z_p^*

Subgroups

- Let $(G, \circ)$ be a group.
  - $(H, \circ)$ is a subgroup of G if
    - $(H, \circ)$ is a group
    - $H \subseteq G$
  - For example, $H = (\{1,2,4\}, \times)$ is a subgroup of Z_7^*.

- *Lagrange's theorem:*

  If $(G, \circ)$ is finite and $(H, \circ)$ is a subgroup of $(G, \circ)$, then $|H|$ divides $|G|$

  For example: 3|6.

Cyclic Groups

- Exponentiation is repeated application of $\circ$
  - $a^3 = a \circ a \circ a.$
  - $a^0 = 1.$
  - $a^{-x} = (a^{-1})^x$
- A group G is cyclic if there exists a generator g, s.t. $\forall a \in G, \exists i$ s.t. $g^i = a$.
  - I.e., G= <g> = {1, g, g^2, g^3, …}
  - For example $Z_7^* = <3> = \{1,3,2,6,4,5\}$
- Not all $a \in G$ are generators of G, but they all generate a subgroup of $G.$
  - E.g. 2 is not a generator of Z_7^*
- The order of a group element a is the smallest $j>0$ s.t. $a^j=1$
- *Lagrange's theorem* $\Rightarrow$ for $x \in Z_p^*$, $ord(x) \mid p-1.$

Fermat’s theorem

- Corollary of Lagrange’s theorem: if $(G, \circ)$ is a finite group, then $\forall a \in G,\ a^{|G|}=1.$
- Corollary (Fermat’s theorem): $\forall a \in Z_p^*,\ a^{p-1} = 1$ mod $p.$ E.g., for all $\forall a \in Z_7^*$, $a^6=1,\ a^7=a.$
- Computing inverses:
- Given $a \in G,$ how to compute $a^{-1}?$
  - Fermat’s theorem: $a^{-1} = a^{|G|-1}$ $(= a^{p-2}$ in $Z_p^*)$
  - Or, using the extended gcd algorithm (for Z_p* or Z_N*):
    - $gcd(a,p) = 1$
    - $s \cdot a + t \cdot p = 1 \Rightarrow s \cdot a = -t \cdot p + 1 \Rightarrow$ s is a^{-1} *!!*
  - Which is more efficient?

Computing in Z_p^*

- P is a huge prime (1024 bits)
- Easy tasks (measured in bit operations):
  - Adding in O(log p) (linear n the length of p)
  - Multiplying in O($\log^2$ p) (and even in O($\log^{1.7}$ p))
  - Inverting (a to a^{-1}) in O($\log^2$ p)
  - Exponentiations:
    - x^r mod p in O(log r · $\log^2$ p), using repeated squaring

Groups we will use

- Z_p^* Multiplication modulo a prime number p
  - $(G, \circ) = (\{1,2,\ldots,p-1\}, \times)$
  - E.g., $Z_7^* = (\{1,2,3,4,5,6\}, \times)$
- Z_N^* Multiplication modulo a composite number N
  - $(G, \circ) = (\{a$ s.t. $1 \leq a \leq N-1$ and $gcd(a,N)=1\}, \times)$
  - E.g., $Z_{10}^* = (\{1,3,7,9\}, \times)$
- A group G is cyclic if there exists a generator g, s.t. $\forall a \in G, \exists i$ s.t. $g^i=a$.
  - I.e., $G = \langle g \rangle = \{1, g, g^2, g^3, \ldots\}$
  - For example $Z_7^* = \langle 3 \rangle = \{1,3,2,6,4,5\}$

Euler's phi function

- Lagrange's Theorem: $\forall a$ in a finite group G, $a^{|G|}=1.$
- Euler's phi function (aka, Euiler's totient function),
  - $\phi(n)$ = number of elements in Z^*_n (i.e. | $\{x$ | $gcd(x,n)=1, 1\leq x\leq n\}$ |
  - $\phi(p) = p-1$ for a prime $p.$
  - $n=\prod_{i=1..k} p_i^{e(i)} \Rightarrow \phi(n) = n\cdot\prod_{i=1..k} (1-1/p_i)$
  - $\phi(p^2) = p(p-1)$ for a prime $p.$
  - $n=p\cdot q \Rightarrow \phi(n) =(p-1)(q-1)$

- Corollary: $\forall a \in Z_n^*$ it holds that $a^{\phi(n)} =1$ mod n
  - For Z_p^* (prime p), $a^{p-1} =1$ mod p (Fermat's theorem).
  - For Z_n^* ($n=p\cdot q$), $a^{(p-1)(q-1)} =1$ mod n

Finding prime numbers

- Prime number theorem: #{primes $\leq x$} ≈ x / ln$x$ as x→∞
- How can we find a random k-bit prime?
  - Choose x at random in $\{2^k,\ldots,2^{k+1}-1\}$
  - Test if x is prime
    - (more on this later in the course)
- The probability of success is ≈ $1/\ln(2^k) = O(1/k)$.
- The expected number of trials is $O(k)$.

Finding generators

- How can we find a generator of Z_p^*?
- Can check whether $\forall\, 1 \le i \le p-2 \quad a^i \neq 1$ ☹
- We know that if $a^i = 1 \bmod p$ then $i \mid p-1$.
- Therefore need to check only i for which $i \mid p-1$.

- Easy if we know the factorization of $(p-1)$
  - For all $a \in Z_p^*$, the order of a divides $(p-1)$
  - For every integer divisor b of $(p-1)$, check if $a^b=1$ mod p.
  - If none of these checks succeeds, then a is a generator.
  - a is a generator iff $ord(a)=p-1.$

Finding prime numbers of the right form

- How can we know the factorization of p-1
- Easy, for example, if *p=2q+1*, and *q* is prime.
- How can we find a *k*-bit prime of this form?

1. Search for a prime number *q* of length *k-1* bits. (Will be successful after about *O(k)* attempts.)
2. Check if *2q+1* is prime.
3. If not, go to step 1.

Quadratic Residues

- The square root of $x \in Z_p^*$ is $y \in Z_p^*$ s.t. $y^2=x$ mod p.
- Examples: sqrt(2) mod 7 = 3, sqrt(3) mod 7 doesn't exist.
- How many square roots does $x \in Z_p^*$ have?
  - If a and b are square roots of x, then $x=a^2=b^2$ mod p. Therefore $(a-b)(a+b)=0$ mod p. Therefore either $a=b$ or $a=-b$ modulo p.
  - Therefore x has either 2 or 0 square roots, and is denoted as a Quadratic Residue (QR) or Non Quadratic Residue (NQR), respectively. How many QRs there are?
- $x^{(p-1)/2}$ is either 1 or -1 in Z_p^* . (indeed, $(x^{(p-1)/2})^2$ is always 1)
- Euler's theorem: $x \in Z_p^*$ is a QR iff $x^{(p-1)/2} = 1$ mod p.
- *Legendre's symbol:*

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & x \text{ is a QR in } Z_p^* \\ -1 & x \text{ is an NQR in } Z_p^* \\ 0 & x = 0 \bmod p \end{cases}$$

- Can be efficiently computed as $x^{(p-1)/2}$ mod p.

Hard problems in cyclic groups

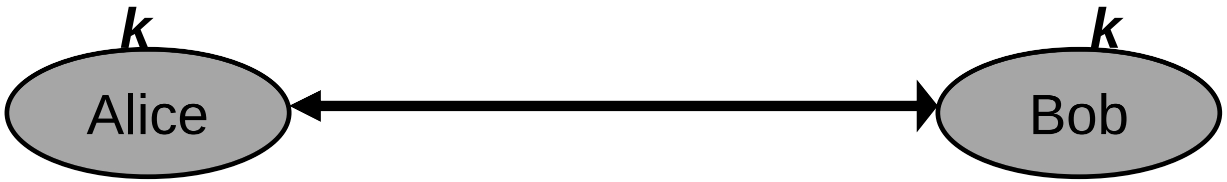
- The following problems are believed to be hard in Z_p^* or in some subgroups of Z_p^*
  - Discrete logarithm: let *g* be a generator of *G*. The input is a random $x \in G$. The task is to find an *r* s.t. $x=g^r \bmod p$.
  - The Diffie-Hellman problem: The input contains *g* and random $x,y \in G$, such that $x=g^a$ and $y=g^b$. The task is to find $z=g^{a \cdot b}$.
  - The Decisional Diffie-Hellman problem: The input contains random $x,y \in G$, such that $x=g^a$ and $y=g^b$; and a pair *(z,z')* where one of *(z,z')* is $g^{a \cdot b}$ and the other is g^c (for a random c). The task is to tell which of *(z,z')* is $g^{a \cdot b}$.
- Solving DDH ≤ solving DH ≤ solving DL
  - All believed to be hard if |p|>1024   (check the next slide)

Does the DDH assumption hold in Z_p^*?

- The DDH assumption does not hold in Z_p^*
  - Assume that both $x=g^a$ and $y=g^b$ are QRs in Z_p^*.
  - Namely, their Legendre symbol is 1, both *a* and *b* are even, and it holds that $x^{(p-1)/2}= y^{(p-1)/2}=1.$
  - Then the Legendre symbol of g^{ab} is always *1,* whereas the symbol of a random g^c is *1* with probability *½.*
- Solution: (work in a subgroup of prime order)
  - Set $p=2q+1$, where *q* is prime.
  - $\phi(Z_p^*) = p-1 = 2q$. Therefore Z_p^* has a subgroup *H* of *prime* order *q*.
  - Let *g* be a generator of *H* *(i.e., g is a QR in Z_p^*).*
  - The DDH assumption is believed to hold in *H.* (The Legendre symbol is always 1.)

Classical symmetric ciphers

- Alice and Bob share a private key *k.*
- System is secure as long as *k* is secret.
- Major problem: generating and distributing *k.*



Diffie and Hellman: “New Directions in Cryptography”, 1976.

- “We stand today on the brink of a revolution in cryptography. The development of cheap digital hardware has freed it from the design limitations of mechanical computing…

  …such applications create a need for new types of cryptographic systems which minimize the necessity of secure key distribution…

  …theoretical developments in information theory and computer science show promise of providing provably secure cryptosystems, changing this ancient art into a science.”

Diffie-Hellman

- Came up with the idea of public key cryptography

public key$_{Bob}$

secret key$_{Bob}$

Alice

Bob

Everyone can learn Bob’s public key and encrypt messages to Bob.
Only Bob knows the decryption key and can decrypt.

Key distribution is greatly simplified.

- Diffie and Hellman did not have an implementation for a public key encryption system
- Suggested a method for key exchange over insecure communication lines, that is still in use today.

Public Key-Exchange

- Goal: Two parties who do not share any secret information, perform a protocol and derive the same shared key.

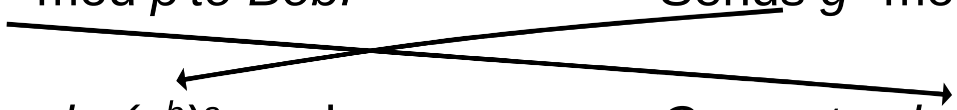
- No eavesdropper can obtain the new shared key (if it has limited computational resources).

- The parties can therefore safely use the key as an encryption key.

The Diffie-Hellman Key Exchange Protocol

- Public parameters: a group Z_p* (where |p|= *768* or *1024, p=2q+1*), and a generator *g* of $H \subset Z_p$* of order *q.*

- Alice:
  - picks a random $a \in [1,q]$.
  - Sends g^a mod p *to Bob.*
- Bob:
  - picks a random $b \in [1,q]$.
  - Sends g^b mod p *to Bob.*



- Alice:
  - *Computes* $k=(g^b)^a$ mod p
- Bob:
  - *Computes* $k=(g^a)^b$ mod p

- $K = g^{ab}$ is used as a shared key between Alice and Bob.
  - DDH assumption ⇒ *K* is indistinguishable from a random key

Diffie-Hellman: security

- A *(passive)* adversary
  - Knows Z_p^*, g
  - Sees g^a, g^b
  - Wants to compute g^{ab}*,* or at least learn something about it
- Recall the Decisional Diffie-Hellman problem:
  - Given random $x,y \in Z_p^*$, such that $x=g^a$ and $y=g^b$*; and a value z* which is either g^{ab} *or* g^c (for a random *c*), it is hard tell which is the case.
  - I.e., g^{ab} is indistinguishable from a random element in *H.*

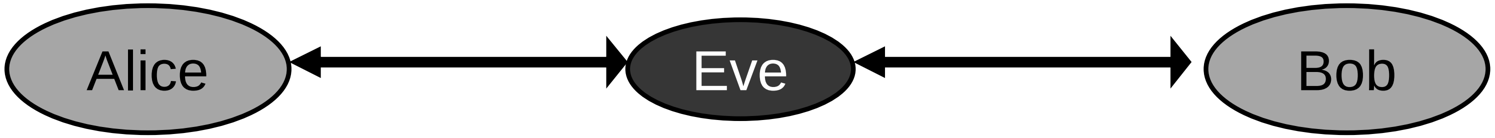
  - *Note:* it is insufficient to require that the adversary cannot compute g^{ab}*.*

Diffie-Hellman key exchange: usage

- The DH key exchange can be used in any group in which the Decisional Diffie-Hellman (DDH) assumption is believed to hold.
- Currently, Z_p* and elliptic curve groups.
- Common usage:
  - Overhead: 1-2 exponentiations
  - Usually,
    - A DH key exchange for generating a master key
    - Master key used to encrypt session keys
    - Session key is used to encrypt traffic with a symmetric cryptosystem

An active attack against the Diffie-Hellman Key Exchange Protocol

- An active adversary Eve.
- Can read and change the communication between Alice and Bob.
- …As if Alice and Bob communicate via Eve.



Man –in-the-Middle: an active attack against the Diffie-Hellman Key Exchange protocol

- Alice:
  - picks a random $a \in [1,q]$.
  - Sends g^a mod p *to Bob.*

Eve changes g^a to g^c

- Bob:
  - picks a random $b \in [1,q]$.
  - Sends g^b mod p to Alice.

Eve changes g^b to g^d

- *Computes* $k=(g^d)^a$ mod p
- *Computes* $k=(g^c)^b$ mod p

Keys:

Alice	Eve	Bob
g^{ad}	g^{ad}, g^{bc}	g^{bc}

- Solution: ?