Introduction to Cryptography
Lecture 12

Secret sharing
Electronic cash

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Secret Sharing

- 3-out-of-3 secret sharing:
  - Three parties, A, B and C.
  - Secret *S.*
  - No two parties should know *anything* about *S*, but all three together should be able to retrieve it.
- In other words
  - $A + B + C \Rightarrow S$
  - But,
    - $A + B \nRightarrow S$
    - $A + C \nRightarrow S$
    - $B + C \nRightarrow S$

Secret Sharing

- 3-out-of-3 secret sharing:
- How about the following scheme:
  - Let $S=s_1s_2\ldots s_m$ be the bit representation of S. (m is a multiple of 3)
    - Party A receives $s_1,\ldots,s_{m/3}$.
    - Party B receives $s_{m/3+1},\ldots,s_{2m/3}$.
    - Party C receives $s_{2m/3+1},\ldots,s_m$.
  - All three parties can recover S.

  - Why doesn't this scheme satisfy the definition of secret sharing?
  - Why does each share need to be as long as the secret?

Secret Sharing

- Solution:
  - Define *shares* for A,B,C in the following way
  - (S_A, S_B, S_C) is a random triple, subject to the constraint that
    - $S_A \oplus S_B \oplus S_C = S$
    - *or,* S_A and S_B are random, and $S_C = S_A \oplus S_B \oplus S$.

- What if it is required that any one of the parties should be able to compute *S?*
  - Set $S_A = S_B = S_C = S$

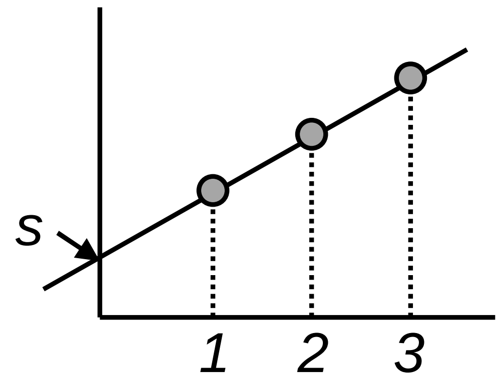
- What if each pair of the three parties should be able to compute *S?*

t-out-of-*n* secret sharing

- Provide shares to *n* parties, satisfying
  - Recoverability: any *t* shares enable the reconstruction of the secret.
  - Secrecy: any *t-1* shares reveal nothing about the secret.

- We saw *1*-out-of-*n* and *n*-out-of-*n* secret sharing.

- Consider *2*-out-of-*n* secret sharing.
  - Define a line which intersects the Y axis at *S*
  - The shares are points on the line
  - Any two shares define *S*
  - A single share reveals nothing



t-out-of-*n* secret sharing

- Fact: Let F be a field. Any $d+1$ pairs (a_i , b_i) define a unique polynomial P of degree ≤ d, s.t. $P(a_i)=b_i$. (assuming $d < |F|$).

- Shamir's secret sharing scheme:
  - Choose a large prime and work in the field *Zp*.
  - The secret S is an element in the field.
  - Define a polynomial P of degree $t-1$ by choosing random coefficients $a_1,\ldots,a_{t-1}$ and defining
    $P(x) = a_{t-1}x^{t-1}+\ldots+a_1x+\underline{S}$.
  - The share of party j is $(j, P(j))$.

t-out-of-n secret sharing

- Reconstruction of the secret:
  - Assume we have $P(x_1),\ldots,P(x_t)$.
  - Use Lagrange interpolation to compute the unique polynomial of degree ≤ $t-1$ which agrees with these points.
  - Output the free coefficient of this polynomial.

- Lagrange interpolation
  - $P(x) = \sum_{i=1..t} P(x_i)\cdot L_i(x)$
  - where $L_i(x)=\prod_{j\neq i}(x-x_j) / \prod_{j\neq i}(x_i-x_j)$
  - (Note that $L_i(x_i)=1$, $L_i(x_j)=0$ for $j\neq i$.)

  - I.e., $S = \sum_{i=1..t} P(x_i) \cdot \prod_{j\neq I} x_j / \prod_{j\neq i}(x_i - x_j)$

Properties of Shamir's secret sharing

- Perfect secrecy: Any *t-1* shares give no information about the secret: $\Pr(secret=s \mid P(1),\ldots,P(t-1)) = \Pr(secret=s)$. (Security is not based on any assumptions.)
- Proof: (Intuition: think about 2-out-of-n secret sharing)
  - The polynomial is generated by choosing a random polynomial of degree *t-1,* subject to *P(0)*=secret.
  - Suppose that the shares are $P(x_1),\ldots,P(x_{t-1})$.
  - *P()* is generated by choosing uniformly random values to the *t-1* coefficients, $a_1,\ldots,a_{t-1}$. (a_0 is already set to be *S*)
    - The values of $P(x_1),\ldots,P(x_{t-1})$ are defined by *t-1* linear equations of $a_1,\ldots,a_{t-1}$, *s.*
    - Since $a_1,\ldots,a_{t-1}$ *are* uniformly distributed, so are the values of $P(x_1),\ldots,P(x_{t-1})$.

Additional properties of Shamir's secret sharing

- Ideal size: Each share is the same size as the secret.
- Extendable: Additional shares can be easily added.
- Flexible: different weights can be given to different parties by giving them more shares.
- Homomorphic property: Suppose $P(1),\ldots,P(n)$ are shares of S, and $P'(1),\ldots,P'(n)$ are shares of S', then $P(1)+P'(1),\ldots,P(n)+P'(n)$ are shares for $S+S'$.

General secret sharing

- P is the set of users (say, n users).
- $A \in \{1,2,\ldots,n\}$ is an authorized subset if it is authorized to access the secret.
- Γ is the set of authorized subsets.
- For example,
  - $P = \{1,2,3,4\}$
  - Γ = *Any set containing one of* $\{ \{1,2,4\}, \{1,3,4,\}, \{2,3\} \}$
  - Not supported by threshold secret sharing

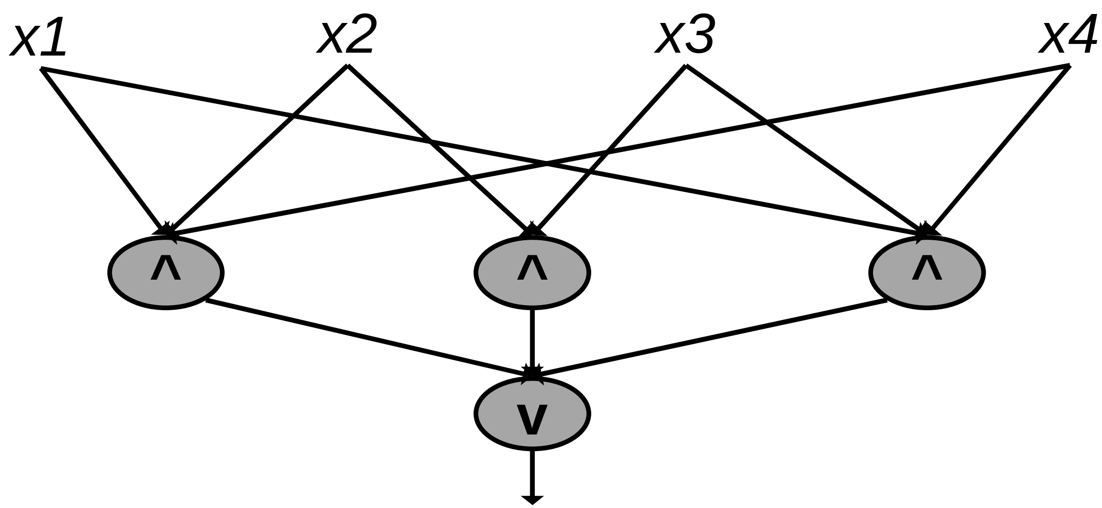
- If $A \in \Gamma$ and $A \subseteq B$, then $B \in \Gamma$.
- $A \in \Gamma$ is a minimal authorized set if there is no $C \subseteq A$ such that $C \in \Gamma$.
- The set of minimal subsets Γ_0 is called the basis of Γ.

Why should we examine general access structures?

- Not all access structures can be represented by threshold access structures
- For example, consider the access structure Γ*={{1,2},{3,4}}*
  - Any threshold based secret sharing scheme with threshold t gives weights to parties, such that $w_1+w_2 \geq t$, and $w_3+w_4 \geq t$.
  - Therefore either $w_1 \geq t/2$, or $w_2 \geq t/2$. Suppose that this is w_1.
  - Similarly either $w_3 \geq t/2$, or $w_4 \geq t/2$. Suppose that this is w_3.
  - In this case parties 1 and 3 can reveal the secret, since $w_1+w_3 \geq t$.
  - Therefore, this access structure cannot be realized by a threshold scheme.

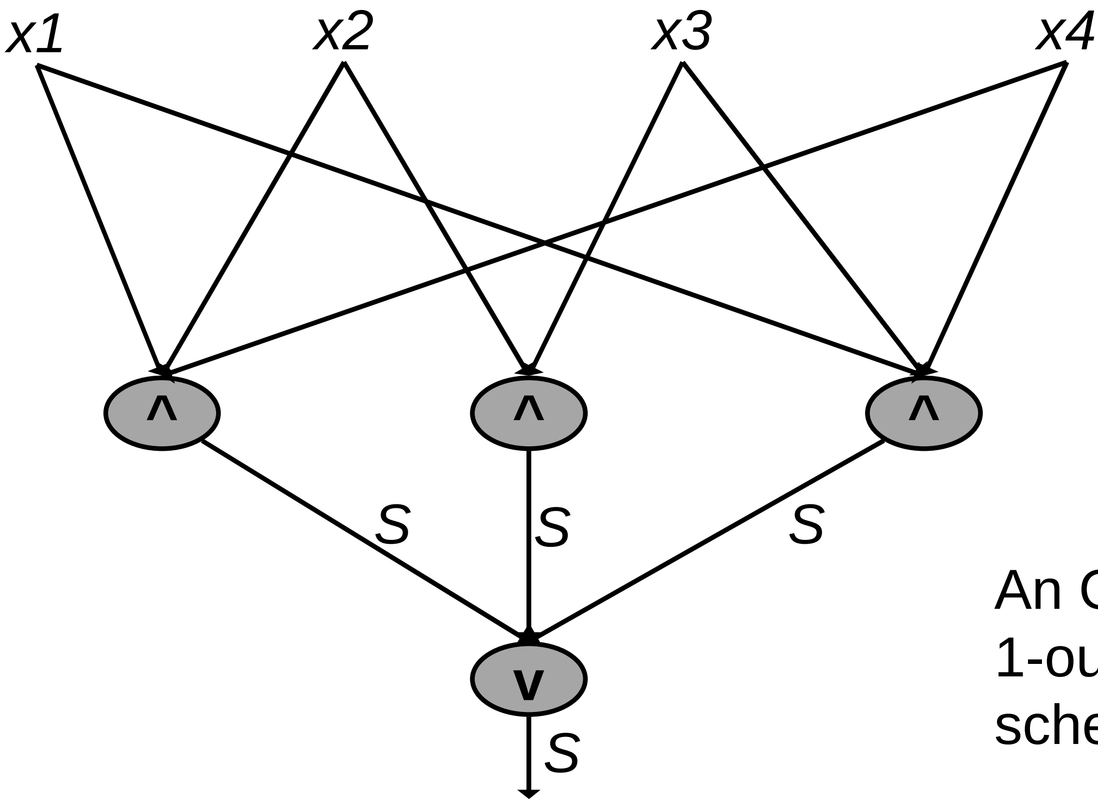
The monotone circuit construction (Benaloh-Leichter)

- A Boolean circuit C with OR and AND gates, is *monotone.* Namely, if *C(x)=1*, then changing bits of *x* from 0 to 1 does not change the result to 0.
- Given Γ construct a circuit C s.t. *C(A)=1* iff $A \in \Gamma$.
  - $\Gamma_0 = \{ \{1,2,4\}, \{1,3,4,\}, \{2,3\} \}$



Handling OR gates

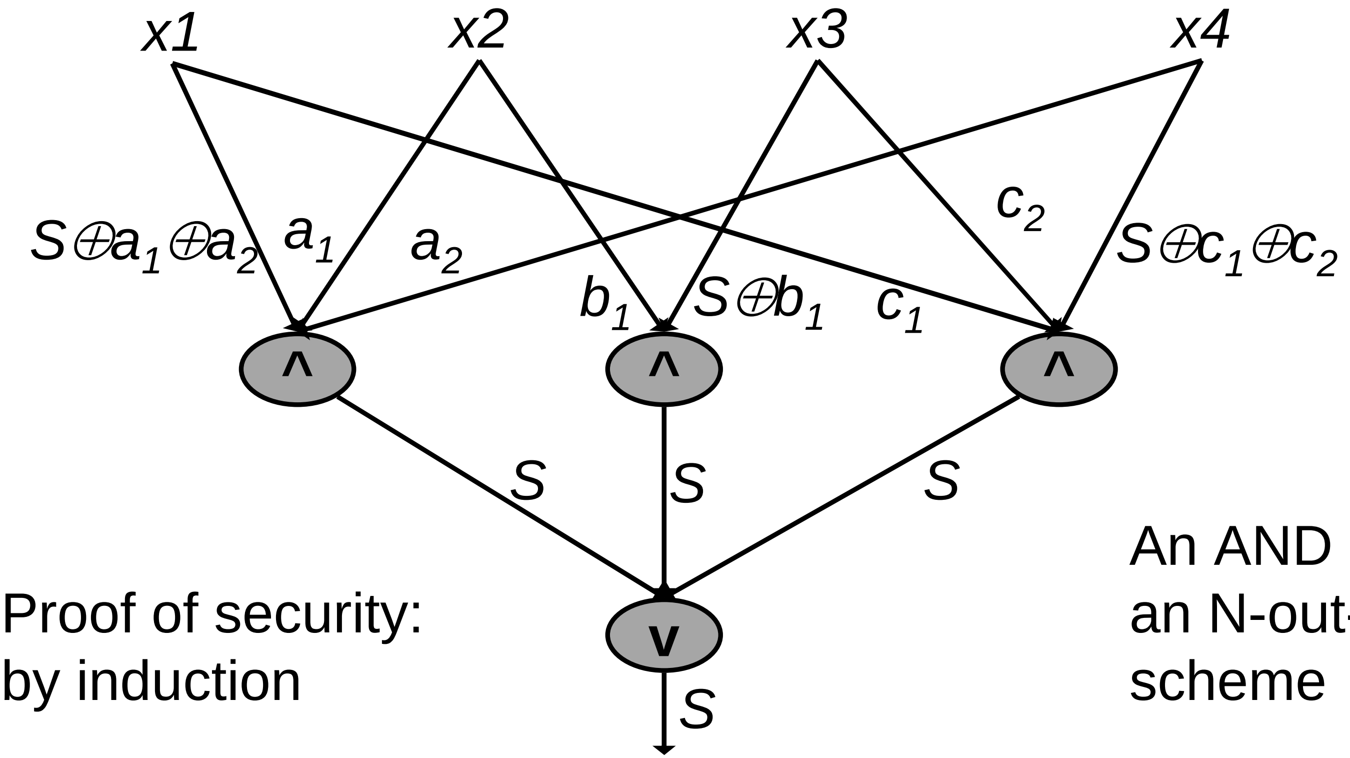
Starting from the output gate and going backwards



An OR gate is a 1-out-of-N scheme

Handling AND gates

Final step: each user gets the keys of the wires going out from its variable

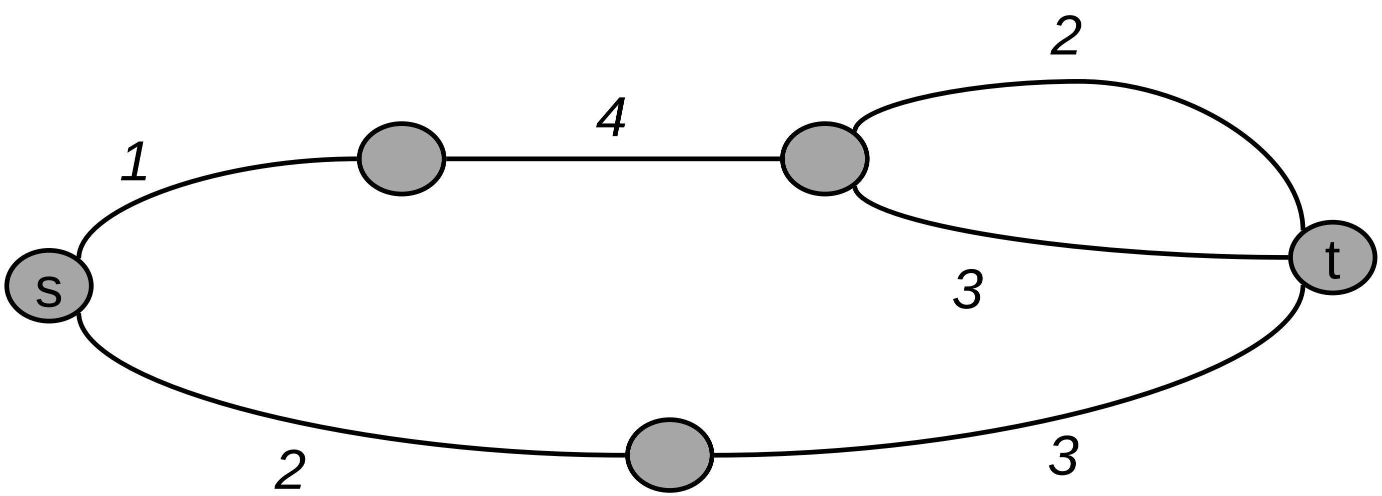


Proof of security: by induction

An AND gate is an N-out-of-N scheme

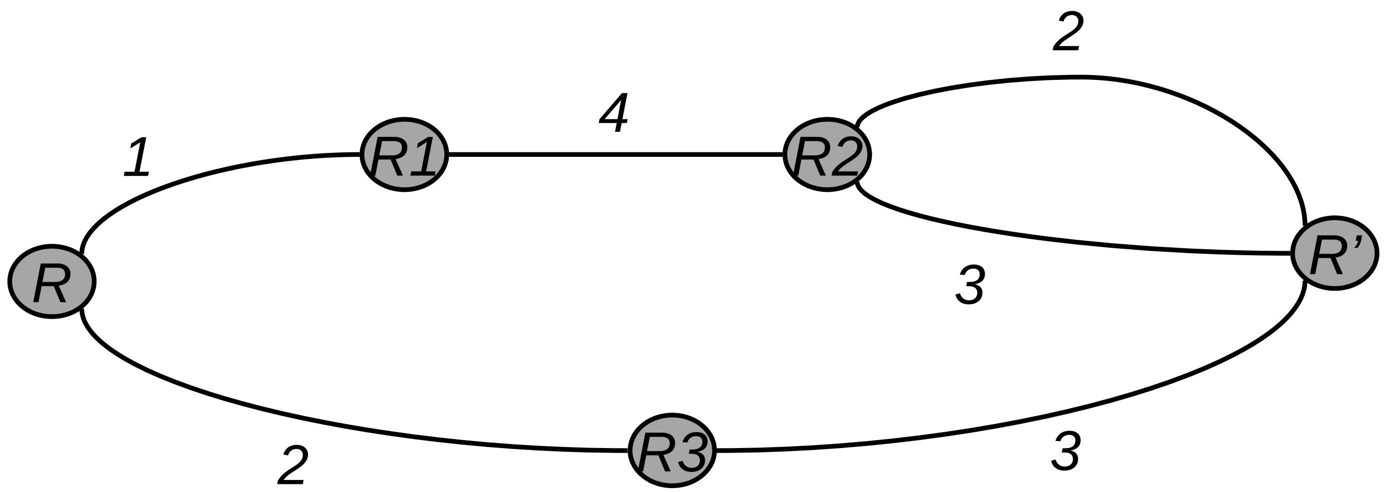
A graph based construction

- Represent the access structure by an undirected graph.
- An authorized set corresponds to a path from s to t in an undirected graph.
- $\Gamma_0 = \{\ \{1,2,4\}, \{1,3,4,\}, \{2,3\}\ \}$

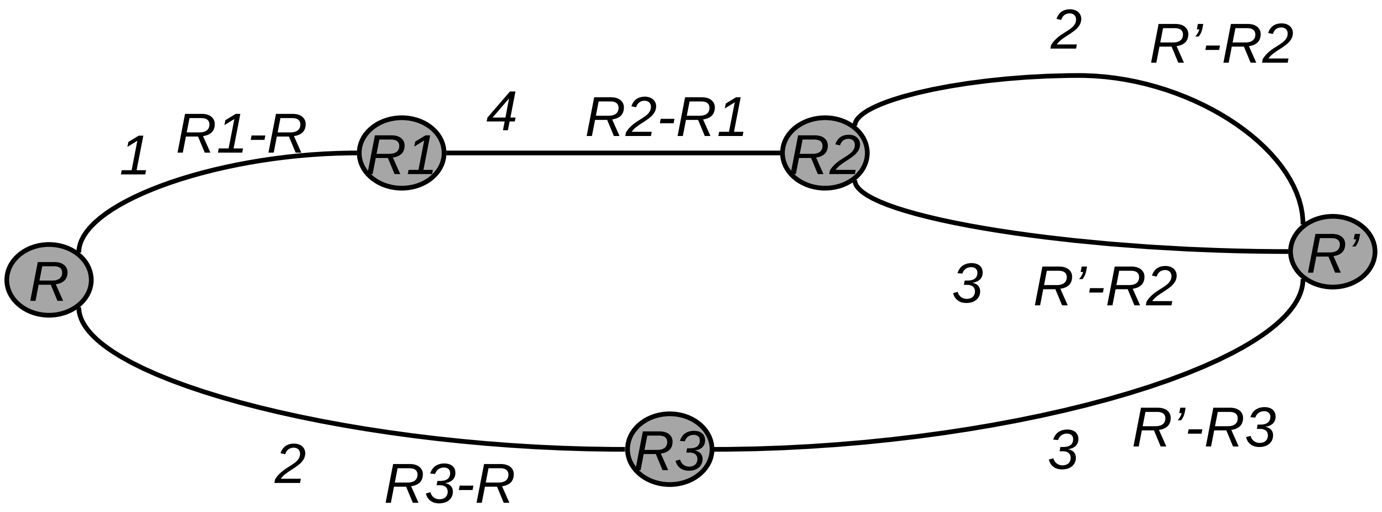


A graph based construction

Assign random values to nodes, s.t. *R'-R= shared secret* *(R'=R+shared secret)*



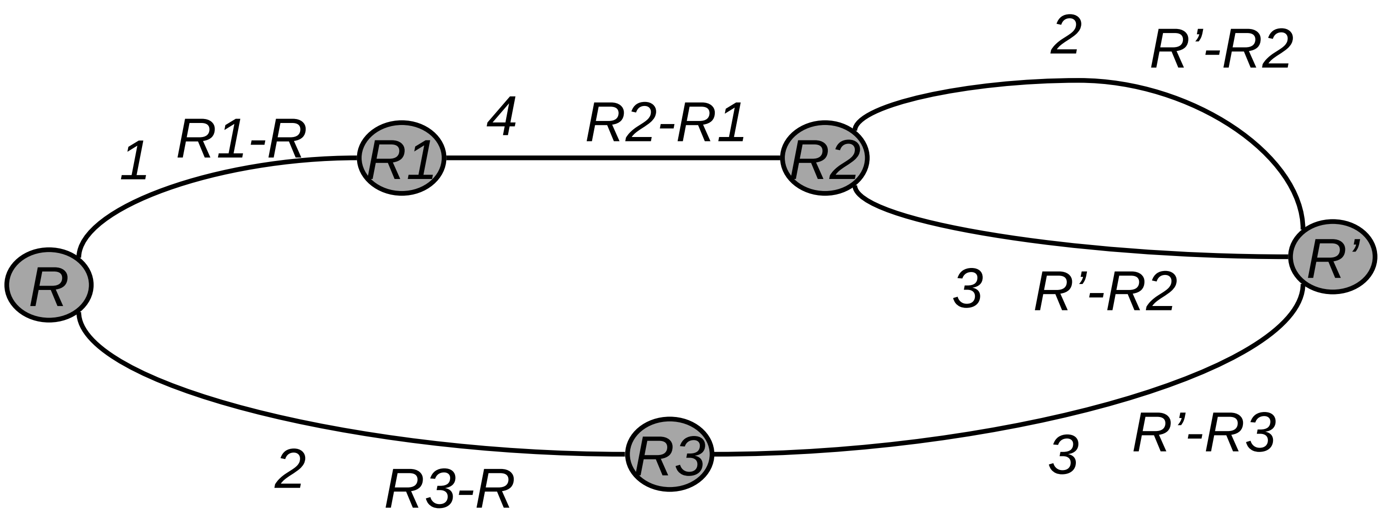
A graph based construction



- Assign to edge R1→R2 the value *R2-R1*
- Give to each user the values associated with its edges

A graph based construction

- Consider the set {1,2,4}
- why can an authorized set reconstruct the secret? Why can't a unauthorized set do that?



Electronic cash

Simple electronic checks

- A payment protocol:
  - Sign a document transferring money from your account to another account
  - This document goes to your bank
  - The bank verifies that this is not a copy of a previous check
  - The bank checks your balance
  - The bank transfers the sum

- Problems:
  - Requires online access to the bank (to prevent reusage)
  - Expensive.
  - The transaction is traceable (namely, the bank knows about the transaction between you and Alice).

First try at a payment protocol

- Withdrawal
  - User gets bank signature on {I am a $100 bill, #1234}
  - Bank deducts $100 from user's account
- Payment
  - User gives the signature to a merchant
  - Merchant verifies the signature, and checks online with the bank to verify that this is the first time that it is used.
- Problems:
  - As before, online access to the bank, and lack of anonymity.
- Advantage:
  - The bank doesn't have to check online whether there is money in the user's account.
  - In fact, there is no real need for the signature, since the bank checks its own signature.

Anonymous cash via blind signatures

- In order to preserve payer's anonymity the bank signs the bill without seeing it
  - (e.g. like signing on a carbon paper)

- RSA Blind signatures (Chaum)
- RSA signature: $(H(m))^{1/e}$ mod n
- Blind RSA signature:
  - Alice sends Bob ($r^e H(m)$) mod n, where r is a random value.
  - Bob computes $(r^e H(m))^{1/e} = r\, H(m)^{1/e}$ mod n, and sends to Alice.
  - Alice divides by r and computes $H(m)^{1/e}$ mod n

- Problem: Alice can get Bob to sign anything, Bob does not know what he is signing.

Enabling the bank to verify the signed value

- "cut and choose" protocol
- Suppose Alice wants to sign a $20 bill.
  - A $20 bill is defined as *H(random index,$20)*.
  - Alice sends to bank 100 different $20 bills for blind signature.
  - The bank chooses 99 of these and asks Alice to unblind them (divide by the corresponding *r* values). It verifies that they are all $20 bills.
  - The bank blindly signs the remaining bill and gives it to Alice.
  - Alice can use the bill without being identified by the bank.

- If Alice tries to cheat she is caught with probability 99/100.
- 100 can be replaced by any parameter *m*.
- But we would like to have an exponentially small cheating probability.

Exponentially small cheating probability

- Define that a \$20 bill is valid if it is the e^{th} root of the multiplication of 50 values of the form *H(x),* (where *x="random index,\$20"*), and the owner of the bill can present all 50 *x* values.

- The withdrawal protocol:
  - Alice sends to the Bank z_1, z_2, …, z_{100} (where $z_i = r_i^{\,e} \cdot H(x_i)$).
  - The Bank asks Alice to reveal ½ of the values $z_i = r_i^{\,e} \cdot H(x_i)$.
  - The Bank verifies them and extracts the e^{th} root of the multiplication of all the other 50 values.
- Payment: Alice sends the signed bill and reveals the 50 preimage values. The merchant sends them to the bank which verifies that they haven't been used before.

- Alice can only cheat if she guesses the 50 locations in which she will be asked to unblind the z_is, which happens with probability $\sim 2^{-100}$.

Online vs. offline digital cash

- We solved the anonymity problem, while verifying that Alice can only get signatures on bills of the right value.

- The bills can still be duplicated
- Merchants must check with the bank whenever they get a new bill, to verify that it wasn't used before.

- A new idea:
  - During the payment protocol the user is forced to encode a random identity string (RIS) into the bill
  - If the bill is used twice, the RIS reveals the user's identity and she loses her anonymity.

Offline digital cash

Withdrawal protocol:

- Alice prepares 100 bills of the form
  - {I am a \$20 bill, *#1234*, $y_1,y'_1,y_2,y'_2,\ldots,y_m,y'_m$}
  - S.t. $\forall i\ y_i=H(x_i)$, $y'_i=H(x'_i)$, and it holds that $x_i \oplus x'_i = Alice's\ id$, where $H()$ is a collision resistant function.
- Alice blinds these bills and sends to the bank.
- The bank asks her to unblind 99 bills and show their x_i, x'_i values, and checks their validity. (Alternatively, as in the previous example, Alice can do a check with fails with only an exponential probability.)
- The bank signs the remaining blinded bill.

Offline digital cash

Payment protocol:

- Alice gives a signed bill to the vendor
  - {I am a $20 bill, #*1234*, $y_1, y'_1, y_2, y'_2, \ldots, y_m, y'_m$}
- The vendor verifies the signature, and if it is valid sends to Alice a random bit string b=$b_1 b_2 \ldots b_m$ of length *m*.
- $\forall i$ if $b_i=0$ *Alice returns* x_i, otherwise $(b_i=1)$ she returns x'_I
- The vendor checks that $y_i=H(x_i)$ or $y'_i=H(x'_i)$ (depending on b_i). If this check is successful it accepts the bill. (Note that Alice's identity is kept secret.)

- Note that the merchant does not need to contact the bank during the payment protocol.

Offline digital cash

- The merchant must deposit the bill in the bank. It cannot use the bill to pay someone else.
  - Because it cannot answer challenges b^* different than the challenge b it sent to Alice.

- How can the bank detect double spenders?
  - Suppose two merchants M and M^* receive the same bill
  - With very high probability, they ask Alice *different* queries b,b^*
  - There is an index i for which $b_i=0,\ b^*_i=1.$ Therefore M receives x_i and M^* receives x'_i.
  - When they deposit the bills, the bank receives x_i and x^*_i, and can compute $x_i \oplus x'_i$ *=Alice's id.*